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ON THE FORMATION OF SHOCK-WAVES IN SUPERSONIC GAS JETS

(TWO-DIMENSIONAL FLOW)

By D. C. PACK (*University College, Dundee*)

[Received 8 September 1947]

SUMMARY

When a jet of gas issues from an orifice as a parallel stream with a given supersonic velocity and flows in a steady state through an outer medium at rest, its behaviour is governed by the ratio between the exit pressure of the jet and the pressure of the outer medium. If this ratio is only a little greater than unity, the jet has a periodic structure to a first approximation. This state has been examined by earlier workers; it is discussed here from the point of view of the 'characteristics' of the hyperbolic second-order partial differential equation of potential flow. The periodic structure ceases to give an adequate representation of the jet as the pressure ratio is increased, and shock-waves occur on account of the compressive effect of the outer medium. A method is given for computing the conditions in a steady two-dimensional supersonic jet. It is shown how the point of origin of a shock-wave and the shape of the shock-wave formation may be obtained by theoretical means. The results of two calculations, given graphically, are discussed and compared, as far as possible, with experimental work.

An expression is obtained for the minimum pressure in a jet.

From the general behaviour of a jet with increasing chamber pressure it is found possible to infer the initial direction of the shock-wave arising at or near the muzzle of a gun after firing.

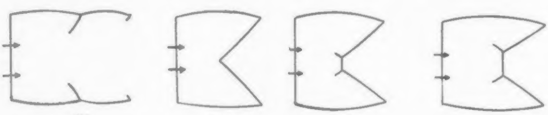
1. Introduction

THERE are many physical problems which depend for their proper resolution upon a knowledge of the behaviour of a jet issuing from an orifice with a velocity greater than the local velocity of sound, two of the most important arising in connexion with the gas stream from a gun and from the exhaust of a rocket motor. These problems are at best rotationally symmetrical, and involve, in their theoretical solution, a great amount of labour in computation, since analytical solutions proper to the initial conditions are not known at present, and numerical (or graphical) methods must be employed. The same difficulty arises when two-dimensional (plane) jets are considered, but the numerical work is shorter; we shall limit ourselves in the present paper to a study of properties of such jets, and in particular to those properties which are the most interesting features of the natural problems—namely, the formation and the form of the shock-waves which occur.

The existing literature on the theory of supersonic gas jets is concerned

with the case in which the pressure of the gas streaming out of the orifice is only a little greater than the pressure in the outside medium; the solutions of Prandtl (**5**) represent perturbations about a steady parallel jet issuing with pressure equal to that of the outer medium, and the jet structure is periodic to a first approximation. As the exit pressure is raised the approximations break down, and a more exact examination of the gas flow is required.

On the experimental side, Prandtl (**6**) took photographs of the plane jet issuing from a nozzle designed to give an approximately parallel exit stream, for a range of ratios of chamber pressure p_0 to outside pressure Π



(i) (ii) (iii) (iv)

FIG. 1.

varying from $p_0/\Pi = 4{\cdot}16$ to $p_0/\Pi = 8{\cdot}13$; the velocity of efflux, using the one-dimensional theory (Reynolds, **7**) for the flow in the nozzle of throat width 7·6 mm., exit width 11·4 mm., was equal to 1·85 times the local velocity of sound. Hartmann and Lazarus (**3**) carried out experiments with a circular nozzle for pressure ratios in the range **3** to 7, the velocity of efflux being equal to the local velocity of sound. We note that there are three parameters for each jet problem for a given gas and adiabatic flow from a reservoir; (1) a scale factor, since jets may be scaled linearly with respect to x and y as the width of the orifice is changed; (2) the Mach number of efflux; (3) the ratio of exit to outside pressure. The experimental information at present available does not represent a systematic investigation into the effect of the variation of the last two of these parameters.

Although shock-waves are inconspicuous in Prandtl's photographs, they are already strikingly in evidence in those of Hartmann and Lazarus for pressure ratios lying well within the range used by Prandtl for his plane jets. As p_0/Π increases (for the cylindrical jet) a short wave occurs in the form of a truncated cone without ends, with its rim at some distance from the axis of the jet (Fig. 1 (i)); the rim moves inwards until a full cone is formed (Fig. 1 (ii)); at still higher pressures a 'Mach' or 'bridge' wave is observed (Fig. 1 (iii) and (iv)).

In the present paper the possibility of accounting theoretically for the occurrence of shock-waves and of calculating their position has been investigated by the solution of two problems in two-dimensional flow using

the method of characteristics. The details of the calculation are given in subsequent sections, together with a graphical representation of the propagation of the characteristics ('wavelets') through the flow, and the positions of the shock-waves. The waves of rarefaction which set in at the bounding edges of the orifice are reflected from the jet boundary as waves of compression, and the diagrams show how this compression involves a sharply steepening pressure gradient; the shock-wave begins where the gradient becomes infinite.

The exit velocity of the jet in each case corresponded to a Mach number of 1·5, and the ratios of chamber pressure to external pressure were 7·1 and 20 respectively. The former, lying within the range of Prandtl's experiments, gave rise to a shock-wave of type corresponding to that in Fig. 1 (i); the latter resulted in a shock-wave which began as in Fig. 1 (iv), but which crossed the axis instead of giving a Mach wave. There seems no reason to suppose that such a wave might not be observed experimentally under appropriate conditions; the pressure ratio was well above those used in the experiments quoted.

2. The equations governing the flow

In order to examine the properties of the gas motion, we consider the jet to have reached a steady state, and to be moving irrotationally, so that a velocity potential ϕ exists. This means that we neglect any transfer of vorticity which may occur as the result of the discontinuity of velocity which exists at the boundary between the jet and the outer medium.

Let axes of coordinates x, y be taken in a normal cross-section of the jet, the x-axis lying along the axis of the jet, and the y-axis perpendicularly to it. Let the origin be taken in the orifice (Fig. 2). Let p_0, ρ_0, a_0 be the pressure, density, and velocity of sound, respectively, of the gas at rest in the discharging vessel (reservoir); write $a_0 u$, $a_0 v$ for the velocity components in the (x, y)-directions respectively, and $a_0 a$ for the local velocity of sound, so that u, v, a are non-dimensional quantities.

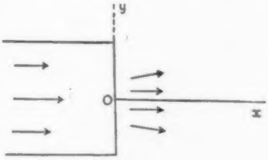


FIG. 2.

If $q^2 = u^2+v^2$, so that q is the non-dimensional resultant velocity, and if θ denotes the angle between the direction of motion at any point and the x-axis, then $\tan\theta = v/u$, and we may define the Mach angle $m = \sin^{-1}(a/q)$.

Let the gas obey the adiabatic law so that the density ρ and the pressure p are connected by the equation $p\rho^{-\gamma} = \text{constant}$, where γ is the ratio of the specific heats. If we write $P = p/p_0$, making P a non-dimensional pressure symbol, then $P = a^{2\gamma/(\gamma-1)}$.

The equation for the velocity potential ϕ is

$$(u^2-a^2)\frac{\partial^2\phi}{\partial x^2}+2uv\frac{\partial^2\phi}{\partial x\partial y}+(v^2-a^2)\frac{\partial^2\phi}{\partial y^2}=0 \tag{1}$$

for two-dimensional flow; since the stream-lines issue from a region of constant conditions (the discharging vessel), Bernoulli's equation possesses a universal constant for the flow, and gives the relation between the gas velocity q and the sound velocity a:

$$q^2 = 2(1-a^2)/(\gamma-1). \tag{2}$$

If the half-width of the jet be taken as the unit length, it is clear from (1) that the results may be scaled by multiplying x and y by any one and the same factor.

The gas flow may be followed by considering the changes which occur along the 'characteristics' (see, for example, Goursat, **2**) of the differential equation (1). The differential equation of the characteristics is

$$(u^2-a^2)\mu^2-2uv\mu+(v^2-a^2)=0, \tag{3}$$

where $\mu = dy/dx$.

At supersonic velocities the characteristics are real, giving two families of lines; one characteristic of each family passes through each point of the flow.

If the roots of (3) are μ_1 and μ_2, it is found that:

$$\left.\begin{array}{l}\text{along the family of characteristic lines } \dfrac{dy}{dx}=\mu_1,\\ \qquad\qquad\qquad\qquad\text{we have } du+\mu_2\,dv=0\\ \text{and along the family of characteristic lines } \dfrac{dy}{dx}=\mu_2,\\ \qquad\qquad\qquad\qquad\text{we have } du+\mu_1\,dv=0\end{array}\right\}. \tag{4}$$

It is easy to show that the stream-line through a point bisects the angle between the characteristics through the point, and that the characteristics make the Mach angle m with the direction of flow.

Using the definition of m and the equation (2), we may write $q = q(m)$, and hence we have

$$u = q(m)\cos\theta, \qquad v = q(m)\sin\theta,$$
$$\mu_1 = \tan(\theta+m), \qquad \mu_2 = \tan(\theta-m).$$

Changing the variables in the relations (4) to θ, m, we obtain integral relations along the characteristics:

$$\text{along } \frac{dy}{dx} = \tan(\theta+m), \quad \mathfrak{p}(m)+\theta = \text{constant}, \tag{5}$$

$$\text{and along } \frac{dy}{dx} = \tan(\theta-m), \quad \mathfrak{p}(m)-\theta = \text{constant}, \tag{6}$$

where $\mathfrak{p}(m) = \dfrac{1}{\lambda}\tan^{-1}\!\left(\dfrac{1}{\lambda}\tan m\right)-m$, and $\lambda^2 = (\gamma-1)/(\gamma+1)$.

It should, perhaps, be pointed out that the expressions on the left-hand sides of these conditions are constant, in general, only along individual members of the family. An exception occurs when a set of characteristics issues from a region of uniform flow (or of rest). Relations similar to (5) and (6) were first found by Steichen (**8**) in his dissertation on two-dimensional gas motion.

3. The properties of a plane supersonic gas jet

The behaviour of the gas jet issuing from an orifice at supersonic velocity into a medium of lower pressure, in the absence of shock-waves, may be followed from Fig. 3, bearing in mind the relations along the characteristics. Let M, M' be the corners of the (section of the) orifice, p_1 the

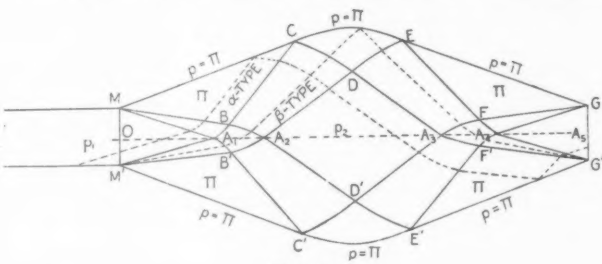


FIG. 3.

pressure across MM', and Π the external pressure. The stream flows unchanged in the region MA_1M', in which the characteristics form two sets of parallel lines making the Mach angle with the axis of flow. We shall refer to these characteristics as being of α-type; coming from a region of uniform axial flow, they possess the same constant in (5) and (6). Around M and M' the gas expands freely in the regions MA_1B, $M'A_1B'$, and then flows in parallel lines in the regions MBC, $M'B'C'$ at pressure Π. Characteristics spring from the corners in the two regions of expansion; these we call characteristics of β-type, each possessing its own particular 'constant', defined by equation (5) or (6) together with the well-known expansion round a corner ('Meyer expansion'—see, for example, Taylor and Maccoll, **9**). Since there is a universal constant for either set of α-characteristics, it follows that when such a set is met by a β-characteristic, there is a unique θ and a unique m holding. Consequently, under such circumstances, a β-characteristic is straight, and the α-characteristics intersect it at a constant angle. This is the case in the regions of the Meyer expansion MA_1B, $M'A_1B'$, and in the regions BA_2DC, $B'A_2D'C'$. It should be noted that each family of characteristics contains characteristics of both α- and β-types.

Across A_2D, A_2D' α-characteristics pass, making equal and opposite angles with the axis of the jet. It is easily seen that this implies a rhomboidal region A_2DA_3D' in which the pressure is uniform (equal to p_2, say), and in which the gas streams parallel to the axis.

The jet boundary is a sheet of velocity discontinuity, with pressure constant across it, and it reflects a characteristic into the other family (see Fig. 3), the angle of reflection being equal to the angle of incidence since the boundary is a stream-line.

The flow may be solved analytically except where β-type characteristics intersect, e.g. A_1BA_2, CDE. In such regions numerical methods must be employed. This also entails numerical methods for the determination of points on (for example) CD, which are obtained from BA_2 via the straight-line β-characteristics in BA_2DC, using conditions of similarity.

From the plane DD' the earlier part of the phenomenon is repeated but with the stages in the reverse order, the expansion regions becoming regions of compression. The β-characteristics reflected from CE, $C'E'$ are straight in the compression regions $DEFA_3$, $D'E'F'A_3$ and the compression waves interfere in the region A_3FA_4F'. The conditions in EFG, $E'F'G'$ are uniform at pressure Π, EG and $E'G'$ being the stream-lines which form the boundary of the jet.

In the Prandtl-Busemann solution of the jet problem (see Busemann, **1**), the β-characteristics, after traversing the interference region A_3FA_4F', become straight on meeting the α-characteristics reflected from EG or $E'G'$, and meet in a point at G (for one set) and G' (for the other set). The whole phenomenon is then repeated with periodic structure, the wave-length being OA_5. This implies the reversibility of the flow; thus, the flow must be isentropic, and no shock-waves occur. In practice, however, as we shall show, β-characteristics of the same family intersect, and, as has been indicated by Taylor and Maccoll (**9**), this may be interpreted to imply the existence of a shock-wave. For ratios of exit to outer pressure close to unity the Prandtl-Busemann solution is, nevertheless, an extremely good approximation, because, in this case, the shock-waves are formed in the neighbourhood of G and G', and are not strong; and it may be shown that, for weak shock-waves, the change in the value of p/ρ^γ is of the third order in the density change—that is to say, the flow is reversible to the third order of small quantities.

The creation of a curved shock-wave in the interior of a fluid involves the breakdown of the irrotational nature of the flow. The increase in entropy in crossing the wave is accompanied by the formation of vorticity behind it. This appears at first sight to contradict the principle of conservation of irrotational flow, but the latter holds only so long as the flow

remains continuous, and it is well known that a compressive motion cannot usually be maintained for more than a finite length of time without the occurrence of a discontinuity.

4. The choice of theoretical problems

In order to see to what extent it might be possible to account theoretically for the occurrence and the form of the shock-waves in a supersonic gas jet, it was necessary to choose for consideration problems which would not involve a prohibitive amount of calculation, but which might be expected to provide interesting illustrations. The exit velocity of the parallel stream at the orifice was taken to correspond to a Mach number of 1·5—when the exit velocity is sonic or near-sonic, difficulties and inaccuracies arise near the orifice on account of the steep gradients of the characteristics—the ratio of specific heats was taken to be 1·3 (corresponding approximately to the gases issuing from the muzzle of a gun), and it was left to decide the chamber pressure. For the first problem the ratio of exit pressure to external pressure (p_1/Π) was taken to be 2, corresponding to a chamber pressure of $7{\cdot}1\Pi$; this served to establish the computational procedure. For the second problem a *chamber* pressure of 20Π was chosen ($p_1 = 7{\cdot}5\Pi$), in order to seek a different type of shock-wave formation.

5. The method of computation

Using the non-dimensional quantities defined earlier, and a suffix π to represent the state in the uniform regions MBC, $M'B'C'$, the conditions required to begin the calculations are as set out in the following table:

Problem	a_1	$u_1 = q_1$	θ_1^0	m_1^0	P_1	a_π	q_π	θ_π^0	m_π^0	P_π
I	0·864675	1·297013	0	41·810	0·283612	0·798213	1·555326	14·382	30·878	0·141806
II	,,	,,	,,	,,	,,	0·707752	1·824074	32·627	22·830	0·050000

It is seen at once, by evaluating the differences of $(\theta - m)$, that the angles of expansion A_1MB, $A_1M'B'$ are equal to 25·314 degrees and 51·607 degrees—and by evaluating the differences of $(\theta + m)$ it is seen that the characteristics $M'A_1B$, MA_1B' are deflected through angles of 3·450 degrees and 13·647 degrees in traversing the region of expansion—in the two cases respectively.

The characteristic $M'A_1B$ in the expansion region A_1B follows the curve $r^2(\sin\beta)^{1/\lambda^2}\cos\beta = \text{const.}$, where

$$\beta = \tan^{-1}\left(\frac{1}{\lambda}\tan m\right) \quad \text{and} \quad \lambda^2 = (\gamma - 1)/(\gamma + 1)$$

as before. The equation may be derived from first principles, using the conditions of the Meyer expansion; r represents the distance from

the appropriate corner. The constant of the α-characteristics is to be determined, and using this in conjunction with the equation of the characteristic found above, values of x, y, θ, and m are calculated at points on $A_1 B$ corresponding to chosen intervals of the angle of expansion, the β-characteristics in this region being straight lines through the corner. We make use of the axis of symmetry, so that only one-half of the field has to be investigated; we restrict ourselves to the upper half. Given the values defining the state at a number of points on $A_1 B$, conditions in $A_1 B A_2$ are calculated, using the method of characteristics. The field has to be built up step by step, the conditions at two points T and B leading to the knowledge of the state at a new point P which is the intersection of the characteristics shown in Fig. 4.

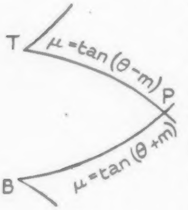


FIG. 4.

From (5) and (6), using suffixes to denote the points to which particular quantities are referred, we have for the point P:

$$\theta_P = \tfrac{1}{2}(\theta_T+\theta_B)-\tfrac{1}{2}(\mathfrak{p}_T-\mathfrak{p}_B), \tag{7}$$

$$\mathfrak{p}_P = \tfrac{1}{2}(\mathfrak{p}_T+\mathfrak{p}_B)-\tfrac{1}{2}(\theta_T-\theta_B). \tag{8}$$

The value of m_P is deduced from (8) by the aid of previously prepared tables of $\mathfrak{p}(m)$ as a function of m. When the point P lies on the axis of x, the formulae reduce to

$$\theta_P = 0, \tag{7a}$$

$$\mathfrak{p}_P = \mathfrak{p}_T-\theta_T. \tag{8a}$$

These formulae are exact. It is necessary, however, to introduce an approximation in order to obtain the coordinates of P. The differential equations for the characteristics are replaced by finite difference relations, and we use, with a slight change of notation, the formulae previously established by Hill and Pack (**4**). If we write $\overset{+}{\mu}$ for the gradient of a characteristic of the family $\mu = \tan(\theta+m)$, and $\overset{-}{\mu}$ for the gradient of a characteristic of the family $\mu = \tan(\theta-m)$, always attaching the appropriate suffix, then putting $\overset{+}{\mu}_P+\overset{+}{\mu}_B = \overset{+}{\Sigma}$, $\overset{-}{\mu}_P+\overset{-}{\mu}_T = \overset{-}{\Sigma}$, and replacing all gradients by the mean of the gradients at the ends of the arc under consideration, the coordinates x, y of P may be expressed by the equations

$$\left.\begin{aligned}(\overset{+}{\Sigma}-\overset{-}{\Sigma})x_P &= \overset{+}{\Sigma}x_B+(-\overset{-}{\Sigma})x_T+2(y_T-y_B)\\ (\overset{+}{\Sigma}-\overset{-}{\Sigma})y_P &= \overset{+}{\Sigma}y_T+(-\overset{-}{\Sigma})y_B+(-\tfrac{1}{2}\overset{+}{\Sigma}\,\overset{-}{\Sigma})(x_T-x_B)\end{aligned}\right\}. \tag{9}$$

When P is an axial point we have

$$\left.\begin{aligned} x_P &= x_T + \frac{2y_T}{(-\bar{\Sigma})} \\ y_P &= 0 \end{aligned}\right\}. \tag{9a}$$

Using the above equations, the region A_1BA_2 is completed, and it becomes necessary to calculate points on CD corresponding to those found on BA_2, remembering that the β-characteristics are straight in $BCDA_2$, and that the α-characteristics cross them at constant angles along their lengths.

We may calculate the coordinates (X_0, Y_0) of C, since we know the equations of MC and BC; with these and the coordinates of two successive points $B(x_0, y_0)$ and $B_1(x_1, y_1)$ on BA_2, we may find the coordinates (X_1, Y_1) of the point C_1 where the β-characteristic through B_1 meets CD. In general (Fig. 5), knowing the coordinates of $C_n(X_n, Y_n)$ on CD, and the consecutive points $B_n(x_n, y_n)$ and $B_{n+1}(x_{n+1}, y_{n+1})$ on BA_2, we may calculate the coordinates of $C_{n+1}(X_{n+1}, Y_{n+1})$ by means of the formulae:

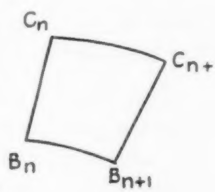


FIG. 5.

$$Y_{n+1} - Y_n = (X_{n+1} - X_n)(y_{n+1} - y_n)/(x_{n+1} - x_n), \tag{10}$$

$$Y_{n+1} - y_{n+1} = (X_{n+1} - x_{n+1})\overset{+}{\mu}_{n+1}, \tag{11}$$

where $\overset{+}{\mu}_{n+1}$ is the gradient of the upward β-characteristic through B_{n+1}. (11) is the equation of this characteristic, and (10) is a line $C_n C_{n+1}$ parallel to $B_n B_{n+1}$, the true gradients of these lines being replaced by the mean value of the gradients at the extremities.

The solution of (10) and (11) gives

$$X_{n+1} = X_n + \frac{(Y_n - \overset{+}{\mu}_{n+1} X_n) - (y_{n+1} - \overset{+}{\mu}_{n+1} x_{n+1})}{\overset{+}{\mu}_{n+1} + \frac{1}{2}(-\bar{\Sigma})}, \tag{12}$$

$$Y_{n+1} = Y_n - \frac{\frac{1}{2}(-\bar{\Sigma})\{(Y_n - \overset{+}{\mu}_{n+1} X_n) - (y_{n+1} - \overset{+}{\mu}_{n+1} x_{n+1})\}}{\overset{+}{\mu}_{n+1} + \frac{1}{2}(-\bar{\Sigma})}. \tag{13}$$

Having obtained conditions at points along CD, we proceed to compute values of θ, m for points in CDE, taking into account the fact that the pressure is constant on the boundary. Constant pressure implies constant m. The coordinates of the points on the boundary may be calculated using a slight modification of (9).

Let θ_n, θ_{n+1} be the directions of the stream at two successive points which we seek on the boundary (Fig. 6). Let the coordinates of the nth boundary point be used as those of T in the earlier work. Put

$$\sigma = \tan\theta_n + \tan\theta_{n+1}$$

(θ_{n+1} is known, since m is given and we know the θ, m relation on the characteristic BP).

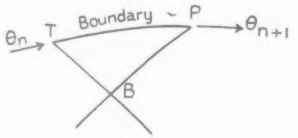


FIG. 6.

The formulae for the coordinates of the $(n+1)$th boundary point P are then identical with those of (9), provided we replace $\bar{\Sigma}$ everywhere by σ. The boundary curves after the manner of CE in Fig. 3.

On completion of this part of the work, the state of the flow is known at a number of points on DE, and these serve to give the field in the region $DEFA_3$, except in so far as this region may be affected by the occurrence of a shock-wave; failing the latter, equations similar to (12) and (13) may be set up to find the flow conditions on $A_3 F$, and the computation continues.

6. The origin of a shock-wave

We have already seen that the effect of the interaction of the expansion waves from the opposite faces of the orifice is to increase the expansion of each. In consequence, the compressions which result from their reflection at the free boundary of the jet are so much the greater, and indeed, the convergence of the β-type characteristics which represent the compression is such that consecutive characteristics eventually intersect one another, and the flow field ceases to be single-valued in the quantities expressing the state at a given point. For $p_0 \sim 7\Pi$, this intersection of characteristics of the same family begins in the region $A_3 FA_4$, after the opposing compressions have experienced some interaction; for $p_0 = 20\Pi$, the intersection occurs between C and D, i.e. before the reflected characteristics reach the axis.

The intersection of characteristics of the same family is to be interpreted as the sign of the breakdown of continuous motion and the beginning of a shock-wave. Taylor and Maccoll (**9**), investigating the flow past a thin curved surface placed edge-on to a supersonic stream, imagine the shock-wave formed as being built up by the intersections of characteristics of the

same family, the shape of the shock-wave, to a first approximation, being the envelope of these characteristics. In fact, the characteristics will only be tangential to the shock-wave at its point of origin, where it is of infinitesimal amplitude. As the strength of the shock-wave increases, the characteristics meet the wave at an angle (see Figs. 7 and 8) determined, along with the position of the shock-wave, by an application of the equations of flow through an oblique discontinuity. This method will be available so long as the flow behind the shock-wave remains supersonic; it will break down at subsonic velocities, because the characteristics then become imaginary.

7. Determination of points on the shock-wave

The shock-wave begins as a wave of infinitesimal amplitude at a point where two characteristics (including CD or A_3F'), whose angles of

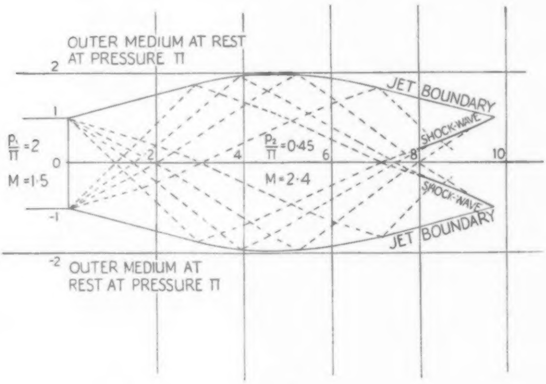


FIG. 7.

departure from M' differ by an infinitesimal quantity, run together tangentially. In order to find this point the field is computed without respect to intersections. The angles of intersection of the 'critical line' (CD or A_3F) and the successive β-characteristics are calculated, and the point at which a characteristic intersects the critical line at zero angle is then computed by the method of divided differences. The initial direction of the shock-wave is the direction of the critical line at this point. The calculated flow downstream of this position must now be abandoned, and the field re-evaluated allowing for the presence of the shock-wave. The results of the two calculations are given graphically in Figs. 7 and 8, showing the shock-waves and some of the characteristics. As the shock-wave curves, the motion behind it ceases to be irrotational, but as the curvature is not

very great, and our primary interest is qualitative, the vorticity has been neglected in the analysis and computation.

The position of any point P on the shock-wave will be determined (i) by the conditions on the characteristics which carry the flow into it; these,

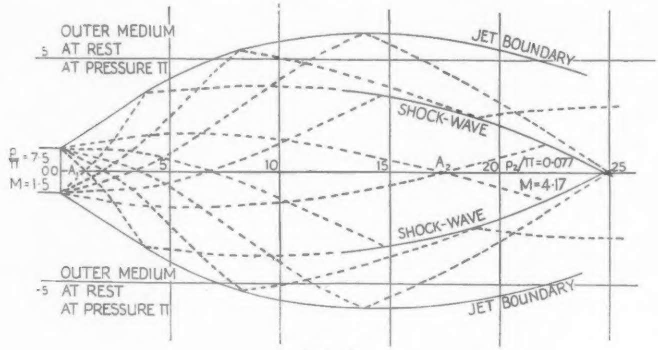


FIG. 8.

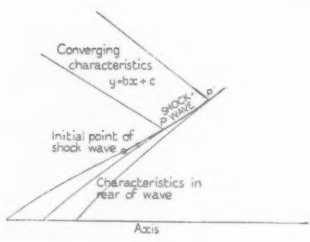


FIG. 9.

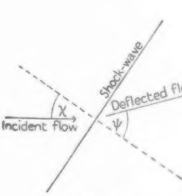


FIG. 10.

being in each case straight lines, can be written in the form $y = bx+c$, and can be represented parametrically by their appropriate value of m; (ii) by the conditions holding on the characteristics which run into what may be termed the rear of the wave (the convex side). The problem thus reduces to the following (Fig. 9):

Given a point P of the shock-wave, we have to determine an arc PP' of it such that the gas flowing into P' is deflected in a manner consistent with the equations for the passage of gas through an oblique shock-wave, and also with the conditions on that characteristic in the rear of the wave which meets the latter at P'.

Let suffixes F, R refer to the front and rear of the shock-wave respectively. Let ψ, χ be the angles between the deflected and the incident flow, respectively, and the normal to the shock-wave at P (Fig. 10).

The relations connecting the conditions on the two sides of the shock-

wave follow from the well-known equations (see, for example, Taylor and Maccoll, **9**). If $z = p_2/p_1$, where p_1 is the pressure in front, and p_2 the pressure in the rear of the shock-wave, then

$$\cos^2\chi = \frac{1}{2\gamma}\{(\gamma-1)+(\gamma+1)z\}\sin^2 m_F,$$

$$\tan\psi/\tan\chi = \{(\gamma-1)+(\gamma+1)z\}/\{(\gamma+1)+(\gamma-1)z\},$$

and
$$\sin m_R = \sin m_F . f(z)\cos\psi/\cos\chi,$$

where
$$f(z) = [z\{(\gamma-1)+(\gamma+1)z\}/\{(\gamma+1)+(\gamma-1)z\}]^{\frac{1}{2}}.$$

By means of the above equations it is possible to construct universal tables for the greater part of the problem; one needs to tabulate against m_F: (i) χ as a function of $(\psi-\chi)$; (ii) m_R as a function of $(\psi-\chi)$. Tables of θ_F, b, c against m_F must be prepared for each problem from the characteristic relationships in front of the wave.

The problem of finding P' is solved, in practice, by moving along a rear characteristic, guessing the value of m_F for the point at which it will meet the shock-wave, thus fixing θ_F, and finding a deflexion of the stream giving values of θ_R and m_R which are consistent with the relation holding on the rear characteristic. For this deflexion we then have a determinable value of the gradient of the shock-wave, and we must next, by an approximation to the equation of the arc of the shock-wave, find its intersection with the forward characteristic (m_F). This gives values for the coordinates of P'; it must be checked if these values are consistent with the (approximate) equation of the rear characteristic—if not, a new m_F must be guessed and the procedure repeated. The method simplifies when the field in front of the shock-wave is uniform, since m_F is fixed. One calculation is sufficient for each point. The method would have increased difficulty in a rotationally symmetrical jet problem, since the forward characteristics would no longer be linear; and indeed, difficulties would arise for the same reason, if we were to try to continue the second of the problems of this investigation beyond the reflection of the shock-wave at the axis.

8. Discussion of results

The accuracy of the solutions obtained is difficult to assess, since the errors involved in each step are additive; these errors, however, arise only in the location of the coordinates. In the present calculations, on the basis of the sizes of the blocks of points requiring numerical evaluation, and of the differences found by doubling the interval between successive points of the network, it seems likely that, beginning with four places of decimals in x, y, the positions are correct to two places of decimals, the

reliability falling off in those parts of the field which are most remote from the orifice, i.e. behind the shock-wave. The results of the two calculations are given graphically in Figs. 7 and 8.

For the jet of Fig. 7 the distribution of pressure along lines parallel to the axis of the jet is given in Fig. 11. It may be seen at once how the pressure falls rapidly in the expansion region to a value p_2 which differs

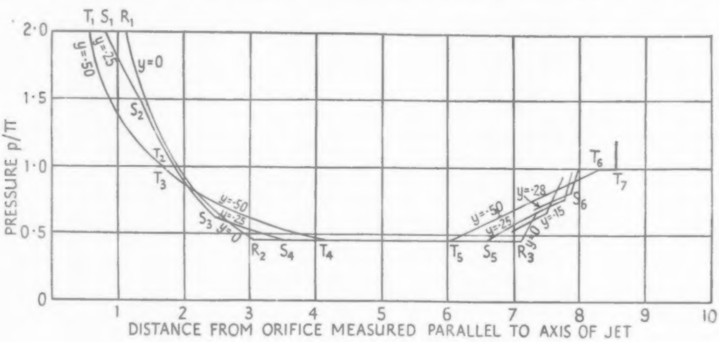


FIG. 11.

but little from Prandtl's approximate value $(\Pi/p_1)\Pi$, derived in the next section. In the region of interference of the two expansion waves there is a sudden increase in the rate of fall of pressure, shown clearly by the discontinuities in gradient on the graphs for $y = 0{\cdot}25$, $y = 0{\cdot}50$, at S_2, T_2 respectively. All the lines along which pressure has been calculated pass through the region of converging straight characteristics, except for the axis which goes directly from the uniform region of pressure p_2 into the part of the field in which the waves of compression reinforce one another. This reinforcement of the compression shows itself for all the lines by a sudden increase in gradient of the pressure graph (R_3, S_6, for example), and the series of graphs drawn illustrates clearly the way in which the shock-wave begins. The gradient discontinuity originating on the axis at R_3 in Fig. 11 (i.e. A_4 in Fig. 3) becomes more and more severe forwards and away from the axis, until eventually it becomes infinite with a resulting instantaneous rise in the magnitude of the pressure itself, i.e. with the formation of a shock-wave. This is almost the case for $y = 0{\cdot}28$ in the figure. For $y = 0{\cdot}5$ there is a finite jump in pressure (at T_7 in Fig. 11), since this line actually passes through the shock-wave itself.

For the 20Π chamber-pressure problem (Fig. 8), the shock-wave curves concavely towards the axis, where it is reflected. The Mach numbers of the flow in front, between, and behind the shock-waves, on the axis, are 4·17, 3·25, and 2·55 respectively. It is to be observed, in this solution,

how sharply the boundary of the jet curves inwards. The jet has almost reached its narrowest section at a point where, with periodic structure, it would be at its maximum width.

From the point of view of comparison with experimental work, it is to be noted that the lesser chamber pressure gives rise to a shock-wave formation of the type shown in Fig. 1 (i), and this accords with expectation, taking into account the difference between the experimental results for two-dimensional and rotationally symmetrical jets. At the higher pressure ratio, the theoretical shock-wave pattern does not conform to the 'Mach' or 'bridge' type, but crosses the axis of symmetry in a regular manner. This may be contrasted with the results of the experiments upon a rotationally symmetrical jet with sonic efflux velocity, for which the Mach wave pattern occurred for all pressure ratios $p_0/\Pi > 4{\cdot}3$.

It is to be noted, finally, that with the increase of chamber pressure, the point of onset of the shock-wave moves backwards along the critical line CD (or $C'D'$) towards C (or C').

9. The minimum pressure

The value of the pressure p_2 in the region A_2DA_3D' (the minimum pressure in the first period) may be deduced from the characteristic relationships. The condition on $M'A_1BC$, namely, $\mathfrak{p}(m)+\theta = \mathfrak{p}(m_1)$, enables the condition on MBA_2 to be written:

$$\mathfrak{p}(m)-\theta = 2\mathfrak{p}(m_\pi)-\mathfrak{p}(m_1),$$

where the suffix unity refers to conditions on efflux, and the suffix π to the uniform regions MBC, $M'B'C'$ in which the pressure is equal to that of the outer medium. If the suffix be used for quantities in the region A_2DA_3D', then putting $\theta = 0$ above,

$$\mathfrak{p}(m_2) = 2\mathfrak{p}(m_\pi)-\mathfrak{p}(m_1). \tag{14}$$

Now let an angle ν be defined by the equation $\tan m = \lambda \tan \nu$. Let $p_1 = n\Pi$; then combining the pressure relationship $p/p_0 = a^{2\gamma/(\gamma-1)}$ with Bernoulli's equation, one obtains

$$p/p_0 = \{2/(\gamma+1)\}^{\gamma/(\gamma-1)}(\sin\nu)^{2\gamma/(\gamma-1)},$$

leading to the two equations which together with (14) determine p_2:

$$\sin\nu_\pi = (1/n)^{(\gamma-1)/2\gamma}\sin\nu_1, \tag{15}$$

$$p_2 = n\Pi\left(\frac{\sin\nu_2}{\sin\nu_1}\right)^{2\gamma/(\gamma-1)} = \Pi\left(\frac{\sin\nu_2}{\sin\nu_\pi}\right)^{2\gamma/(\gamma-1)}. \tag{16}$$

We see that $p_1p_2 = n^2\Pi^2\left(\frac{\sin\nu_2}{\sin\nu_1}\right)^{2\gamma/(\gamma-1)}$. If we put $n = 1+\epsilon$, where ϵ is small compared with unity, it is easy to show that $(\sin\nu_2/\sin\nu_1)^{2\gamma/(\gamma-1)} \sim n^{-2}$. We thus derive the well-known Prandtl rule for the minimum pressure,

$p_1 p_2 \sim \Pi^2$. The attainment of the minimum pressure expected from the equations (14)–(16) is clearly dependent upon the flow in $MOM'A_2$ being totally unaffected by the presence of any shock-wave; shock-waves may freely exist outside this region without affecting the result.

It has already been remarked that the minimum pressure found with $p_1 = 2\Pi$ was in fairly close agreement with that given by Prandtl's rule. For the pressure $p_1 = 7{\cdot}5\Pi$, however, the minimum pressure was calculated to be $0{\cdot}077\Pi$, differing greatly, as we should expect, from the value derived from the simple rule.

10. The flow of gases from the muzzle of a gun

It was noted at the end of § 8 that the effect of an increase in the chamber pressure is to move the point of onset of the shock-wave back along $G'C$ towards C. The point cannot move beyond C because it must come in a region where the motion is one of compression; on the other hand, C can be made to approach M by reducing the velocity of flow of the gas from the orifice. Using these facts as a starting-point we can infer that, for sonic exit velocity and a sufficiently high chamber pressure, the shock-wave may be considered to begin, with zero strength, approximately at the orifice itself.

For a parallel, uniform jet issuing with a pressure in excess of that of the external medium, this seems to be the only way in which a shock-wave can begin at the orifice. For the same jet issuing at under-pressure a shock-wave beginning at the lip of the orifice is to be expected; and it is possible for shock-waves to occur inside the lip of a nozzle for which the design is not sufficiently accurate to ensure a parallel uniform stream.

As a corollary to the above remarks, it is of interest to consider the gas flow issuing from the muzzle of a gun; the flow velocity is approximately sonic. Since in the limit at M the initial directions of the characteristics in the two-dimensional and the rotationally symmetric problems are the same, we may infer, in particular, that the direction of the characteristic CD at C in the plane case, in the limit as $\sin m_1 \to 1$ and C approaches M, gives the *initial* direction, approximately, of the shock-wave from the muzzle of a gun.

Using the notation of the earlier part of the paper, it follows from Bernoulli's equation that

$$\{\tfrac{1}{2}\operatorname{cosec}^2 m_\pi + 1/(\gamma-1)\}\Pi^{(\gamma-1)/\gamma} = \{\tfrac{1}{2}\operatorname{cosec}^2 m_1 + 1/(\gamma-1)\}(n\Pi)^{(\gamma-1)/\gamma}.$$

BC makes an angle $(\theta_\pi + m_\pi)$ with Ox, and it is known that

$$-\theta_\pi = \mathrm{p}(m_\pi) - \mathrm{p}(m_1).$$

The angle between MC and Ox is θ_π, hence the direction at C of CD, the

reflection of BC, is at an angle $(m_\pi - \theta_\pi)$ with Ox, i.e. it makes an angle $\xi = \frac{1}{2}\pi - m_\pi + \theta_\pi$ with the direction of MM', which, by the relation above, gives $\xi = \frac{1}{2}\pi - \mathfrak{p}(m_\pi) - m_\pi + \mathfrak{p}(m_1)$. For sonic efflux, $m_1 = \frac{1}{2}\pi$ and

$$\mathfrak{p}(m_1) = \tfrac{1}{2}\pi(1-\lambda)/\lambda.$$

From Bernoulli's equation, $\tan m_\pi = \lambda/(n^{(\gamma-1)/\gamma} - 1)^{\frac{1}{2}}$. It follows at once that

$$\xi = \frac{1}{\lambda}\tan^{-1}\sqrt{(n^{(\gamma-1)/\gamma} - 1)}. \qquad (17)$$

For $n = \infty$, this gives $\xi = \pi/2\lambda \sim 249°$; the shock-wave begins in a direction inclined steeply backwards from the plane of the orifice.

For $n = 500$, a high muzzle pressure for a gun, $\xi \sim 168°$, i.e. it begins just forward of the continuation of the plane of the muzzle.

Acknowledgements

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A VARIATIONAL PRINCIPLE OF MAXIMUM PLASTIC WORK IN CLASSICAL PLASTICITY

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SUMMARY

The classical equations of Lévy-Mises and Prandtl-Reuss for an ideally plastic material are reviewed. A variational principle of maximum plastic work is derived for plastic states of stress satisfying the Lévy-Mises relation and the Huber-Mises yield criterion. Uniqueness theorems are established for a completely plastic body under prescribed boundary conditions. The variational principle is applied to the problem of a uniform bar of arbitrary section deformed in combined tension, torsion, and bending.

1. The yield criterion and stress-strain relations

THE following considerations are concerned with an ideally plastic material which is perfectly elastic up to a sharp yield point and thereafter does not work-harden. The criterion of yielding under combined stresses σ_{ij} is taken to be that of Huber-Mises:

$$\sigma'_{ij}\sigma'_{ij} = 2k^2, \tag{1}$$

where σ'_{ij} is the deviatoric component of the stress tensor σ_{ij}. σ'_{ij} is also called the reduced stress tensor. k is equal to $Y/\sqrt{3}$, where Y is the yield stress in uniaxial tension or compression. Since the work-hardening is assumed to be zero, k remains constant during the plastic deformation.

The relations between stress and strain-increment for an isotropic element of material which is being plastically deformed are taken to be

$$d\epsilon'_{ij} = \frac{d\sigma'_{ij}}{2G} + \sigma'_{ij}\,d\lambda \tag{2}$$

and

$$d\epsilon_{ii} = \frac{1-2\nu}{E}\,d\sigma_{ii}. \tag{3}$$

E, G, ν are the elastic constants: Young's modulus, shear modulus, and Poisson's ratio respectively. $d\epsilon_{ij}$ is the tensor representing an increment of true or natural strain, measured with respect to the current configuration. $d\lambda$ is a scalar, essentially positive, but otherwise unspecified and generally varying both in time and space. The first term on the right-hand side of equation (2) is the deviatoric component of the increment of elastic strain, also measured with respect to the current configuration. The second term is the increment of plastic or permanent strain. Equation (3) expresses the experimental fact that the elastic compressibility is unchanged by the plastic distortion. It is supposed, further, that all the

elastic constants remain invariable, provided they are defined in terms of the current configuration. These equations are due to Reuss (**13**), who based them on the work of Saint-Venant, Lévy, von Mises, and Prandtl. When the plastic flow is 'free' and takes place under constant stress, the elastic strain-increments are zero and the equations are then equivalent to the Lévy-Mises relations:

$$d\epsilon_{ij} = \sigma'_{ij}\,d\lambda. \tag{4}$$

The criterion of yielding (1) has been shown to be valid for copper, nickel, aluminium, iron, strain-hardened mild steel, medium carbon and alloy steels (Ros and Eichinger, **14**; Taylor and Quinney, **19**; Davis, **1** and **2**; Lessells and MacGregor, **6**). Of course for a real metal k and Y depend on the previous strain-history. The Lévy-Mises relations (4) have been shown to be valid to a first approximation for copper, nickel, iron, aluminium, and strain-hardened mild steel (Lode, **8**; Taylor and Quinney, **19**; Schmidt, **17**; Davis, **2**). When the flow is not free, and the elastic and plastic components of the strain are comparable, there is only limited evidence in support of the Reuss equations (2). The general appropriateness of the Prandtl-Reuss method of allowing for varying elastic strains in the ideally plastic body can, however, hardly be doubted.

2. Stationary work principle and the plastic potential

A method of representing the reduced stress and strain-increment tensors on a plane diagram was introduced by Reuss (**13**). This is particularly useful for a general discussion of stress-strain relationships. The principal components of stress σ_1, σ_2, σ_3 are plotted along Cartesian axes of reference (Fig. 1). $\overrightarrow{OS}$ is the vector $(\sigma_1, \sigma_2, \sigma_3)$ and $\overrightarrow{OP}$ the vector representing the reduced principal stresses $(\sigma'_1, \sigma'_2, \sigma'_3)$. P therefore lies in the plane $\sigma_1+\sigma_2+\sigma_3 = 0$. The vector $\overrightarrow{PS}$ is perpendicular to this plane, and has components (σ, σ, σ), where $\sigma = \frac{1}{3}(\sigma_1+\sigma_2+\sigma_3)$ is the mean of the principal stresses. The principal components $d\epsilon_1$, $d\epsilon_2$, $d\epsilon_3$ of the increment of true strain can be represented by a vector in the same space by plotting $2G\,d\epsilon_1$, etc. The vector representing the reduced strain-increment $(d\epsilon'_1, d\epsilon'_2, d\epsilon'_3)$ will lie in the plane $\sigma_1+\sigma_2+\sigma_3 = 0$. In this way the relations between reduced stress and strain-increment for an isotropic material can be followed on a plane diagram; the hydrostatic component of stress need not be considered in discussing plastic behaviour. The strain-history of an element can be represented by some curved path in the plane, built

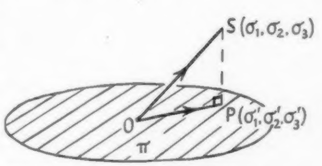

FIG. 1. Reuss's geometrical representation of stress. π is the plane $\sigma_1+\sigma_2+\sigma_3 = 0$.

up of successive vector increments. The 'final' strain will be denoted by the point Q. There is no physical significance in the vector $\overrightarrow{OQ}$, except for purely elastic strains or when the strain path is such that the principal axes of strain-increment do not rotate relatively to the element.

Suppose now that the principal axes of stress *do not rotate* relatively to the element. While the element is still only strained elastically, P and Q coincide. After yielding, the points separate and P moves on a circle, centre O and radius $\sqrt{(\frac{2}{3})}Y$, representing the yield criterion (1) (Fig. 2).

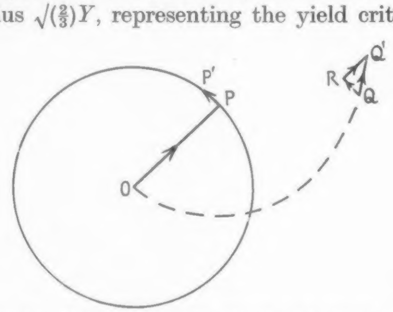

FIG. 2. Huber-Mises circle with stress and strain-increment vectors in plane $\sigma_1+\sigma_2+\sigma_3 = 0$.

Suppose $\overline{QQ'}$ is a strain-increment given to the element following a moment when the stress vector is $\overline{OP}$, and suppose the stress changes to a neighbouring stress vector $\overline{OP'}$ on the circle during this strain. The principal axes of the elastic and plastic strain-increments coincide in view of the initial supposition. According to equation (2) the plastic strain-increment is parallel to $\overline{OP}$, and the elastic strain-increment is parallel to $\overline{PP'}$, that is, perpendicular to $\overline{OP}$. Hence, if $\overline{RQ'}$, $\overline{QR}$ are the components of $\overline{QQ'}$ along and perpendicular to $\overline{OP}$, they represent respectively the plastic and elastic strain-increments. The change in stress $\overline{PP'}$ is thus equal to $\overline{QR}$. The work done on the element is positive so long as $\overline{OP}.\overline{QQ'}$ is positive; $d\lambda$ is then also positive. Otherwise unloading occurs.

Consider, now, free flow where the elastic strain-increment is zero or negligible. We do not need to retain the restriction that the principal stress axes do not rotate. Suppose the strain-increment $\overline{QQ'}$ is prescribed, while no restriction is placed on the stress beyond requiring P to lie on the circle. Then the incremental work done in the strain $\overline{QQ'}$, corresponding to some stress P, is just $\frac{QQ'}{2G}\times$the projection of $\overline{OP}$ along $\overline{QQ'}$. This is a maximum when $\overline{OP}$ is parallel to $\overline{QQ'}$, that is, when the Lévy-Mises relation is satisfied. Thus the Lévy-Mises relation corresponds to maximum plastic work in a prescribed strain-increment for varying stress

subject to the Huber-Mises yield criterion (Hill, Lee, and Tupper, **4**). Von Mises (**10**) stated this merely as a stationary work principle, without examining whether a maximum or minimum was involved. He suggested further that for any yield criterion the appropriate stress-strain relations could be generated on the basis of such a principle; this idea has also been independently proposed by Taylor (**18**). The stress corresponding to a given strain-increment would then be represented by the point on the yield locus where the normal is parallel to the increment of strain vector.

In 1928 von Mises introduced the concept of a plastic potential. If the criterion of yielding is

$$f(\sigma_{ij}) = \text{constant}, \tag{5}$$

the plastic potential is defined to be the function f of the stress components. The definition is not quite unique, but this does not matter in the subsequent application. Von Mises suggested that the stress-strain relations in free flow which are appropriate for a material with this yield criterion should be

$$d\epsilon_{ij} = \frac{\partial f}{\partial \sigma_{ij}}\, d\lambda, \tag{6}$$

where $d\lambda$ is a positive scalar factor of proportionality. Since σ_{ij} is a tensor, so also is $\partial f/\partial \sigma_{ij}$; and if f is invariant in form for an arbitrary choice of axes (as it must be for an isotropic material) then so is $\partial f/\partial \sigma_{ij}$. Hence the above relation is thus far suitable to represent a strain. When f is the Huber-Mises expression $\sigma'_{ij}\sigma'_{ij}$ it is easily verified that equation (6) leads to the Lévy-Mises relations. In general, as von Mises showed, postulating the existence of a plastic potential is equivalent to postulating a stationary work principle. For the condition that $\sigma_{ij}d\epsilon_{ij}$ should be stationary for given $d\epsilon_{ij}$ and varying σ_{ij} subject to $f(\sigma_{ij}) = \text{constant}$, is

$$\frac{\partial}{\partial \sigma_{pq}}\{\sigma_{ij}d\epsilon_{ij} - f\,d\lambda\} = 0,$$

where $d\lambda$ is an undetermined multiplier. This is simply equation (6). It should be noticed that the requirement of the existence of a true stationary value will, for some yield criteria, place restrictions on possible strains. In particular it can be shown that the volume change is zero if, and only if, the yielding does not depend on hydrostatic pressure, so that f is a function only of σ'_{ij}. Thus the plastic potential is not only consistent with the two observed facts of zero plastic volume change, and yielding independent of superposed hydrostatic pressure, but provides a link between them.

It is interesting to examine the plastic potential idea in relation to the behaviour of mild steel. Annealed mild steel yields in accordance with

the maximum shear stress criterion of yielding. The yielding is, moreover, non-uniform and consists of localized simple shears in the Lüders' lines which traverse the specimen in the yield-point extension range of strain. This is exactly the type of deformation corresponding to a potential $f(\sigma_{ij}) \equiv \tau_{\max}$; that is, a shear in the direction of the maximum shear stress. On the other hand, cold-worked mild steel yields uniformly and obeys the Huber-Mises and Lévy-Mises laws which, as we have seen, are also consistent with the existence of a potential.

3. The general principle of maximum plastic work

In the last section a maximum work principle was established for an element in free flow obeying the laws (1) and (4). A general statement of this principle will now be proved for a finite mass of plastic material, no element of which is being unloaded.

Suppose a plastic mass is in quasi-static equilibrium under a stress system σ_{ij}. Let u_i be the flow velocities on the bounding surface of the mass. Velocity in this context is not measured on a time-scale, since indefinitely slow flow is being considered, but relatively to the prescribed quasi-static motion of some part of the system. The system σ_{ij} satisfies the equilibrium equations, the yield criterion (1), and the free-flow relations (4). Then the rate at which the surface forces of this system do work is greater than that for any other system $\bar{\sigma}_{ij}$, for the same surface velocities, provided $\bar{\sigma}_{ij}$ satisfies the equilibrium equations and the yield criterion. The proof is as follows.

The actual rate of work of the external forces

$$
\begin{aligned}
&= \int \sigma_{ij} u_i l_j \, dS \qquad (l_j = \text{direction cosines of the outward normal}) \\
&= \int \frac{\partial}{\partial x_j}(\sigma_{ij} u_i)\, dV \qquad (\text{through the volume}) \\
&= \int \sigma_{ij} \frac{\partial u_i}{\partial x_j}\, dV \qquad (\text{in view of the equilibrium equations}) \\
&= \int \lambda \sigma'_{ij} \sigma'_{ij}\, dV
\end{aligned}
$$

by using equation (4) expressed in terms of velocities, with λ as the new proportionality factor. The rate of work for the $\bar{\sigma}_{ij}$ system with the same surface velocities is

$$\int \bar{\sigma}_{ij} u_i l_j \, dS = \int \frac{\partial}{\partial x_j}(\bar{\sigma}_{ij} u_i)\, dV = \int \bar{\sigma}_{ij} \frac{\partial u_i}{\partial x_j}\, dV$$

(since the system $\bar{\sigma}_{ij}$ is assumed to be in equilibrium)

$$= \int \lambda \bar{\sigma}'_{ij} \sigma'_{ij}\, dV.$$

Now λ is essentially positive, and so the principle is proved if

$$\sigma'_{ij}\sigma'_{ij} \geqslant \bar{\sigma}'_{ij}\sigma'_{ij} \quad \text{whenever} \quad \sigma'_{ij}\sigma'_{ij} = \bar{\sigma}'_{ij}\bar{\sigma}'_{ij}.$$

This is equivalent to the statement that the scalar product of the stress vectors to any two points on the yield circle (Fig. 2) is never greater than the square of the radius, and is clearly true. The equal sign corresponds to systems differing only by a hydrostatic pressure or tension.

This variational principle can be adapted to prove certain uniqueness theorems. If arbitrary velocities are taken over the surface, then there is either no possible corresponding stress distribution for which the *whole* mass is plastic, or, if one exists, it is unique (apart from a uniform hydrostatic pressure). The first part of this statement follows from a consideration of special cases; a general criterion for determining possible surface velocity distributions has not been found. To prove the second, suppose $(\sigma_{ij}, u_i, \lambda)$ and $(\bar{\sigma}_{ij}, \bar{u}_i, \bar{\lambda})$ could be two possible systems such that $u_i = \bar{u}_i$ over the surface. Then

$$\int (\bar{\sigma}_{ij}-\sigma_{ij})u_i l_j\,dS = \int \lambda\sigma'_{ij}(\bar{\sigma}'_{ij}-\sigma'_{ij})\,dV = -\tfrac{1}{2}\int \lambda(\bar{\sigma}'_{ij}-\sigma'_{ij})(\bar{\sigma}'_{ij}-\sigma'_{ij})\,dV,$$

and

$$\int (\sigma_{ij}-\bar{\sigma}_{ij})\bar{u}_i l_j\,dS = \int \bar{\lambda}\bar{\sigma}'_{ij}(\sigma'_{ij}-\bar{\sigma}'_{ij})\,dV = -\tfrac{1}{2}\int \bar{\lambda}(\bar{\sigma}'_{ij}-\sigma'_{ij})(\bar{\sigma}'_{ij}-\sigma'_{ij})\,dV.$$

Since $u_i = \bar{u}_i$ on the surface, we obtain, by addition of these equations,

$$\int (\lambda+\bar{\lambda})(\bar{\sigma}'_{ij}-\sigma'_{ij})(\bar{\sigma}'_{ij}-\sigma'_{ij})\,dV = 0.$$

But $(\lambda+\bar{\lambda})$ is everywhere positive, and $(\bar{\sigma}'_{ij}-\sigma'_{ij})(\bar{\sigma}'_{ij}-\sigma'_{ij}) \geqslant 0$ for all stress systems. Hence $\bar{\sigma}'_{ij} \equiv \sigma'_{ij}$. The theorem can be slightly extended to the case when the velocities are prescribed only over a part of the surface; over the rest certain external components of force are zero, while those velocity components for which the non-zero force components do work are also prescribed.

Similarly, if the stresses are specified over the surface, then either no completely plastic state of stress exists, or, if one exists, it is unique. For, if σ_{ij}, $\bar{\sigma}_{ij}$ could be two possible states, it has been seen that

$$\int (\bar{\sigma}_{ij}-\sigma_{ij})u_i l_j\,dS = -\tfrac{1}{2}\int \lambda(\bar{\sigma}'_{ij}-\sigma'_{ij})(\bar{\sigma}'_{ij}-\sigma'_{ij})\,dV.$$

Hence, if $l_j\bar{\sigma}_{ij} = l_j\sigma_{ij}$ on the surface, both sides of the equation are zero, and the proof goes as before. The velocities are not, of course, uniquely determined. A more general statement can be proved for the case when the external forces are prescribed only over a part of the surface, some of the velocity components being zero over the rest; in this part the external force components which do work for the non-zero velocity components are also given.

It is interesting to consider the maximum work principle in relation to a familiar energy theorem in elasticity, which is invariably stated in a minimal form. It will now be shown that the two variational principles are special cases of the same, more general, theorem. Suppose two sets of quantities σ_{ij} and u_i are related by the system of equations

$$\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) = \alpha\sigma'_{ij}+\beta\sigma\delta_{ij},$$

where α, β may be variable but are positive, and $\sigma = \frac{1}{3}\sigma_{ii}$. Again σ_{ij} satisfies the equilibrium equations $\dfrac{\partial\sigma_{ij}}{\partial x_j} = 0$. Define

$$I = \tfrac{1}{2}\int (\alpha\sigma'_{ij}\sigma'_{ij}+3\beta\sigma^2)\,dV; \qquad J = \int \sigma_{ij}u_i l_j\,dS,$$

where I is an integral through a volume, and J is an integral over the surface of the volume. Then, if $\bar{\sigma}_{ij}$ satisfies the equilibrium equations,

$$\bar{I}-\bar{J} \geqslant I-J,$$

where $\bar{I}$, $\bar{J}$ denote I, J with $\bar{\sigma}_{ij}$ substituted for σ_{ij} while α, β and the u_i are unaltered. This statement is proved by observing that

$$\bar{I}-I \equiv \int (\bar{\sigma}_{ij}-\sigma_{ij})(\alpha\sigma'_{ij}+\beta\sigma\delta_{ij})\,dV+I(\bar{\sigma}_{ij}-\sigma_{ij}),$$

in which $I(\bar{\sigma}_{ij}-\sigma_{ij})$ denotes that $\bar{\sigma}_{ij}-\sigma_{ij}$ is substituted for σ_{ij} in $I(\sigma_{ij})$. Therefore

$$\bar{I}-I = \int (\bar{\sigma}_{ij}-\sigma_{ij})\frac{\partial u_i}{\partial x_j}\,dV+I(\bar{\sigma}_{ij}-\sigma_{ij})$$

$$= \bar{J}-J+I(\bar{\sigma}_{ij}-\sigma_{ij}),$$

after using the equilibrium equations and Green's theorem. But since α, β are positive, $I(\bar{\sigma}_{ij}-\sigma_{ij})$ is also positive for all σ_{ij}, $\bar{\sigma}_{ij}$, and the result follows. By setting $\alpha = \dfrac{1}{2G}$, $\beta = \dfrac{1-2\nu}{E}$, and interpreting u_i as the total displacement we recover the familiar energy theorem in elasticity. This is usually stated in the form: *if the boundary displacements are given, then of all statically possible stress distributions the actual one makes $I-J$ a minimum.* In the actual solution $I = \frac{1}{2}J =$ potential elastic energy. If, on the other hand, we set $\alpha = \lambda$, $\beta = 0$, and interpret the u_i as velocities, we recover the maximum work principle in plasticity. For $I = \bar{I}$ by the yield criterion, and so $(-J) \leqslant (-\bar{J})$ or $J \geqslant \bar{J}$. There is thus nothing peculiar in having a maximal principle in plasticity and a minimal principle in elasticity.

All the above theorems remain true for a material which work-hardens, provided elastic strain-increments are neglected. The value of k (or Y) is then a function of the previous strain-history, and varies from point

to point of the material. In the statement of the variational principle the stress system $\bar{\sigma}_{ij}$ must be such that $\bar{\sigma}'_{ij}\bar{\sigma}'_{ij}$ is not constant but is equal to the given value of $\sigma'_{ij}\sigma'_{ij}$ at each point at the moment under consideration. Free flow is not possible for a material which hardens, since the stress always changes during the prescribed strain-increment, and the elastic strain-increment is not zero. When hardening occurs, therefore, the theorems are strictly true only for a fictitious plastic-rigid material.

4. Application of the variational principle

The variational principle of maximum plastic work can be used to solve certain special problems. However, its range of application is probably less wide than that for the analogous theorem of elasticity, since the boundary conditions cannot be prescribed arbitrarily if the region under consideration is to be entirely plastic. The principle will be applied here to determine the stress distribution in a prismatic bar of arbitrary uniform cross-section, plastically deformed by certain forces and couples applied to the ends. The cylindrical surface of the bar is supposed to be stress-free, and elastic strains are neglected. Cartesian axes of reference are taken so that the z-axis is parallel to the generators; the axes of x and y lie in the plane of a cross-section. For present purposes it is not necessary to select any special point as the origin of coordinates.

It is first necessary to examine what surface velocities can be prescribed if the bar is to be completely plastic. For simplicity consider only conditions where the stresses and the rates of strain do not depend on z. This rules out the possibility of flexural forces, but allows certain combinations of bending couple, twisting couple, and longitudinal force. Let u, v, w be the component velocities. It is easy to show that, if the material is incompressible and the rates of strain are independent of z, the most general expressions for the velocities (apart from a rigid-body motion) are

$$\left.\begin{aligned} u &= \frac{\partial\phi}{\partial y} - \tfrac{1}{2}A(x^2+z^2) - \tfrac{1}{2}Cx + Dyz; \\ v &= -\frac{\partial\phi}{\partial x} - \tfrac{1}{2}B(y^2+z^2) - \tfrac{1}{2}Cy - Dxz; \\ w &= \psi(x,y) + Axz + Byz + Cz. \end{aligned}\right\} \tag{7}$$

ϕ, ψ are arbitrary functions of x, y; ψ determines the warping of a cross-section. A, B, C, D are dimensional constants related to the rates of bending, extension, and torsion. Of these possible velocity distributions only a special sub-set represents the deformation of a completely plastic bar; this will be examined subsequently.

These velocities will now be used in the variational principle. In addition to the assumption that the stresses do not depend on z, it is supposed

that σ_{xx}, σ_{yy}, σ_{xy} are identically zero, by analogy with the corresponding problem in elasticity. Such assumptions are always necessary to obtain a tractable variational problem; their validity must be verified afterwards by substitution in the equations (1) and (4). In view of the equilibrium equations we can introduce a stress function $f(x,y)$ such that

$$\sigma_{xz} = -k\frac{\partial f}{\partial y}; \qquad \sigma_{yz} = k\frac{\partial f}{\partial x}; \tag{8}$$

where $f = 0$ on the contour of the section since there are no external forces on the cylindrical surface. The longitudinal stress σ_{zz} is given by the yield condition:

$$\sigma_{zz} = \sqrt{3}\,k(1-f_x^2-f_y^2)^{\frac{1}{2}}, \tag{9}$$

where f_x, f_y are written for $\partial f/\partial x$, $\partial f/\partial y$. The rate of work of the external forces is then

$$\dot{W} = \iint (\sigma_{xz}\,u+\sigma_{yz}\,v+\sigma_{zz}\,w)\,dxdy, \tag{10}$$

where the integral is taken over the ends of the bar. This expression has to be maximized for surface values of u, v, w (tentatively assumed to be given by equations (7)) with respect to stresses satisfying equations (8) and (9). It is not necessary to know in advance the actual velocities on the cylindrical surface since the external forces there are zero, and contribute nothing to the work done.

In equation (10) terms not involving z cancel out at the opposite ends of the bar. If L is the length of the bar, the remaining terms lead to

$$\frac{\dot{W}}{kL} = \sqrt{3}\iint (Ax+By+C)(1-f_x^2-f_y^2)^{\frac{1}{2}}\,dxdy - D\iint (xf_x+yf_y)\,dxdy + \\ +\tfrac{1}{2}L\iint (Af_y-Bf_x)\,dxdy.$$

On the right-hand side the third term is zero since $f = 0$ on the contour, and the second term can be transformed to give

$$\frac{\dot{W}}{kL} = \sqrt{3}\iint (Ax+By+C)(1-f_x^2-f_y^2)^{\frac{1}{2}}\,dxdy + 2D\iint f\,dxdy. \tag{11}$$

Stationary values of $\dot{W}$ then correspond to functions f satisfying

$$\frac{\partial}{\partial x}\left[\frac{(Ax+By+C)f_x}{\sqrt{(1-f_x^2-f_y^2)}}\right] + \frac{\partial}{\partial y}\left[\frac{(Ax+By+C)f_y}{\sqrt{(1-f_x^2-f_y^2)}}\right] + \frac{2D}{\sqrt{3}} = 0. \tag{12}$$

The solution of this equation, subject to the boundary condition $f = 0$, determines the stress function f and thereby the stresses σ_{xz}, σ_{yz}, σ_{zz}. The equation is elliptic, as can be seen by writing it in the form

$$(Ax+By+C)\{(1-f_y^2)f_{xx}+2f_xf_yf_{xy}+(1-f_x^2)f_{yy}\}+ \\ +(Af_x+Bf_y)(1-f_x^2-f_y^2)+\frac{2D}{\sqrt{3}}(1-f_x^2-f_y^2)^{\frac{3}{2}} = 0. \tag{13}$$

The particular integral $1-f_x^2-f_y^2=0$ corresponds to the case of pure torsion. Generally it will probably be more convenient to apply Rayleigh-Ritz methods in the variational problem of equation (11), rather than solve equation (13) directly.

We have now to verify that a corresponding velocity solution can be found such that the Lévy-Mises equations are satisfied. By substituting from equations (7), (8), and (9) in equation (4), it is readily proved that

$$\phi(x,y) = \tfrac{1}{12}(Ay^3-Bx^3)+\tfrac{1}{4}xy(Ax-By); \tag{14}$$

$$\frac{\partial\psi}{\partial x} = -Dy-\frac{\sqrt{3}(Ax+By+C)f_y}{\sqrt{(1-f_x^2-f_y^2)}}; \tag{15}$$

$$\frac{\partial\psi}{\partial y} = Dx+\frac{\sqrt{3}(Ax+By+C)f_x}{\sqrt{(1-f_x^2-f_y^2)}}. \tag{16}$$

The condition for the compatibility of equations (15) and (16) for the warping function ψ is just equation (12). Substituting the expression for ϕ into equation (7), the velocities for a completely plastic state are

$$\left.\begin{aligned} u &= \tfrac{1}{4}A(y^2-x^2-2z^2)-\tfrac{1}{2}Bxy-\tfrac{1}{2}Cx+Dyz;\\ v &= -\tfrac{1}{2}Axy+\tfrac{1}{4}B(x^2-y^2-2z^2)-\tfrac{1}{2}Cy-Dxz;\\ w &= \psi(x,y)+Axz+Byz+Cz.\end{aligned}\right\} \tag{17}$$

The corresponding external forces applied at the ends of the bar are statically equivalent to a longitudinal force

$$Z = \sqrt{3}\,k\iint(1-f_x^2-f_y^2)^{\frac{1}{2}}\,dxdy,$$

a twisting couple $$G_z = 2k\iint f\,dxdy,$$

and a bending couple with components

$$G_x = \sqrt{3}\,k\iint y(1-f_x^2-f_y^2)^{\frac{1}{2}}\,dxdy; \qquad G_y = \sqrt{3}\,k\iint x(1-f_x^2-f_y^2)^{\frac{1}{2}}\,dxdy.$$

It should be noted that the sign of the radical is not necessarily the same over the whole cross-section. To given values of the three ratios A/D, B/D, C/D, expressing the relative rates of bending, twisting, and extension, correspond values of Z, G_x, G_y, G_z. On the other hand, three ratios of the latter cannot be arbitrarily prescribed if the whole bar is to be plastic; for example, a single bending couple (G_x, G_y) does not generally produce a completely plastic state.

Finally let us consider the present variational principle of maximum plastic work in relation to the heuristic maximum 'effort' principle proposed by Sadowsky (**15**) for the ideally plastic body. Sadowsky suggested that 'among all statically determined possible stress distributions (satisfying all three equations of equilibrium, the condition of plasticity, and boundary conditions) the actual stress distribution in plastic flow requires

a maximum value of the external effort necessary to maintain the flow'. Sadowsky verified this tentative principle in the case of a uniform bar of circular section, plastically deformed in combined torsion and tension, by maximizing the longitudinal load for a given torque or vice versa. Prager (**11**) showed, more generally, that the principle could be applied to obtain the correct differential equation for a uniform prismatic bar of arbitrary section under combined torsion and tension. Handelman (**3**) verified Sadowsky's principle for a bar under combined bending and torsion, the section having an axis of symmetry, by finding stationary values of the bending moment for a given torque. The reason for the success of Sadowsky's heuristic principle in all these cases (and in certain others) is that the actual velocities turn out to be such that the problems of finding stationary values of the work done, and of the 'effort' (i.e. load, bending moment, torque), are identical. This identity has already been observed in the special case of combined torsion and tension of a circular cylinder (Waters, **20**). Generally, of course, Sadowsky's principle does not lead to correct results, even where the applied forces are such that an unambiguous meaning can be attached to the term 'effort'.

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ON A FUNCTION ASSOCIATED WITH THE LOGARITHMIC DERIVATIVE OF THE GAMMA FUNCTION

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SUMMARY

The function $\psi(k)-\log k+1/2k$ is real for real positive k, and complex for pure imaginary k, its imaginary part being then given by a simple formula. Its real part, considered as a function of $1/k^2$, varies smoothly through $1/k^2 = 0$, and is a convenient auxiliary function to use for interpolation purposes; it is also required in connexion with the tabulation of the confluent hypergeometric function.

The real part of $\psi(k)-\log k+1/2k$ has been evaluated, and is tabulated, as a function of $1/k^2$ for the range $1/k^2$ $-1{\cdot}00(0{\cdot}01)+1{\cdot}00$, to 8 decimals.

1. Introduction

In one group of applications of wave mechanics to the calculation of the structure and properties of many-electron atoms, it is required to have tables of two independent solutions of the equation

$$\frac{d^2y}{dx^2}+\left[\frac{1}{x}-\tfrac{1}{4}\gamma-\frac{l(l+1)}{x^2}\right]y = 0 \tag{1}$$

for integral values of l, γ being a constant which may take both positive and negative values, including zero (**1, 2**). There is a recurrence relation between solutions for three successive integral values of l (**2**), so that it is sufficient to consider the cases $l = 0$ and $l = 1$.

Solutions of equation (1) can be expressed in terms of the confluent hypergeometric function $W_{k,m}(z)$ of Whittaker (see ref. **3**, ch. 16); two solutions are

$$y = W_{k,l+\frac{1}{2}}(x/k) \quad \text{and} \quad y = W_{-k,l+\frac{1}{2}}(-x/k), \tag{2}$$

where $k^2 = 1/\gamma$. But this form is inconvenient in this context, first, because here the variation of the solution with γ at constant x (not at constant $\gamma^{\frac{1}{2}}x$) is important, and this is obscured by writing the solution in the form (2) (see ref. **1**); secondly, the solutions for different values of γ are required as functions of x, not as functions of $\gamma^{\frac{1}{2}}x$, and thirdly, because the form (2) is clearly not appropriate for $\gamma = 0$, and looks likely to be awkward for small positive and for negative values of γ, all of which are important in this application.

Consideration of the convenient solutions of (1) to take in this context (see ref. **1**) leads to the conclusion that in the expansion of one of these

solutions near the origin a coefficient has to be used which involves γ in the combination

$$\psi(1/\gamma^{\frac{1}{2}})+\tfrac{1}{2}\log\gamma+\tfrac{1}{2}\gamma^{\frac{1}{2}}, \tag{3}$$

where $\psi(z)$ is the logarithmic derivative of the Γ-function,

$$\psi(z) = \Gamma'(z)/\Gamma(z)$$

(the notation used here is that of Whittaker and Watson (**3**), chapter 12); for positive values of γ, the positive value of $\gamma^{\frac{1}{2}}$ is to be understood. It is convenient to write

$$k^2 = 1/\gamma \tag{4}$$

with the same understanding Then the quantity (3) is

$$\psi(k)-\log k+1/2k. \tag{5}$$

The values of the function (5) are required, as a function of $1/k^2$, for real positive and negative values of this argument.

For γ positive (k real) the function (5) is real; for γ negative (k pure imaginary), it is complex. From the asymptotic formula for the Γ-function, which holds provided $|\arg z| < \frac{1}{2}\pi$, it follows that

$$\psi(z)-\log z+1/2z \sim \sum_{r=1}^{\infty}(-)^r B_r/2rz^{2r}$$

(the notation for Bernoulli's numbers here is that of Whittaker and Watson (**3**), § 7.2), that is,

$$\psi(z)-\log z+1/2z \sim -\frac{1}{12}\left[\frac{1}{z^2}-\frac{1}{10}\frac{1}{z^4}+\frac{1}{21}\frac{1}{z^6}-\frac{1}{20}\frac{1}{z^8}+\frac{1}{11}\frac{1}{z^{10}}\cdots\right], \tag{6}$$

so that for small positive values of γ, $\psi(k)-\log k+1/2k$ behaves like $-\frac{1}{12}\gamma$, and is zero for $\gamma = 0$.

Formally, the asymptotic series (6) does not apply to pure imaginary values of z (γ negative); but it is found that both the real and imaginary parts of $\psi(k)-\log k+1/2k$, regarded as functions of γ, difference smoothly through $\gamma = 0$. This function is thus a convenient auxiliary function to use for interpolating $\psi(k)$ for real positive values and pure imaginary values of k, and for this reason may have applications beyond those for which it was originally required; it has therefore been calculated and tabulated to a greater accuracy than that required in that context.

2. $\gamma > 0$

For $\gamma > 0$, $k = \gamma^{-\frac{1}{2}}$ is real, and there is then no difficulty in evaluating $\psi(k)-\log k+1/2k$, either from Davis's tables (**4**) of $\psi(x)$ (written $\Psi(x)$ in these tables), or, for sufficiently small γ, from the asymptotic series. To obtain, from Davis's tables, values of $\psi(k)-\log k+1/2k$ for integer values of $100\gamma = 100/k^2$, it is most convenient to evaluate $\psi(x)-\log x+1/2x$ for a small group of values of x, in the neighbourhood of $k = 10/(100\gamma)^{\frac{1}{2}}$, for which $\psi(x)$ can be taken from Davis's tables without interpolation, and which bracket the required value of k, and to interpolate between these values, rather than interpolating between the values of $\psi(x)$ itself.

3. $\gamma < 0$

For $\gamma < 0$, $k = \gamma^{-\frac{1}{2}}$ is pure imaginary, whereas the asymptotic series (6) is only valid for $|\arg z| < \frac{1}{2}\pi$. For pure imaginary values of z, say $z = iy$,

$$|\Gamma(iy)| = (\pi/y \sinh \pi y)^{\frac{1}{2}}$$

(Whittaker and Watson, ref. **3**, ch. 12, Example 7), and hence the imaginary part of $\psi(iy)$ is

$$\operatorname{Im}\psi(iy) = \tfrac{1}{2}\pi \coth \pi y + 1/2y \tag{7}$$

(see ref. **5**, p. 203), so that for $k = i|\gamma^{-\frac{1}{2}}|$,

$$\begin{aligned}\operatorname{Im}[\psi(k) - \log k + 1/2k] &= \pi/(e^{2\pi k} - 1)\\ &= \pi/[\exp(2\pi|\gamma^{-\frac{1}{2}}|) - 1] \quad (\gamma < 0).\end{aligned} \tag{8}$$

This is a function for which all derivatives of finite order are zero at $\gamma = 0$, so that it joins smoothly at $\gamma = 0$ on to

$$\operatorname{Im}[\psi(k) - \log k + 1/2k] = 0 \quad (\gamma > 0).$$

Thus numerical methods are only necessary to evaluate the real part of $\psi(k) - \log k + 1/2k$ for $(1/k^2) < 0$. The method used was to evaluate the real part of $\psi(n + i|k|)$ for $n = 10$, for each negative value of $1/k^2$, from the asymptotic series (6), and then to step down to $n = 0$ using the recurrence relation

$$\psi(z) = \psi(z+1) - 1/z. \tag{9}$$

The required tabular value is

$$\operatorname{Re}[\psi(i|k|) - \log i|k| + 1/2i|k|] = \operatorname{Re}\psi(i|k|) - \log|k|. \tag{10}$$

This was done for each integer value of -50γ from 0 to 50.

If the terms up to that in $1/z^{10}$ in the asymptotic formula for $\psi(n + i|k|)$ are taken, the error in the result is less than 1.10^{-11} for $n = 10$, so that use of these terms is adequate to give 10-place accuracy. The real parts of these terms give

$$\begin{aligned}\operatorname{Re}\psi(n+i|k|) - \log|k| &= \tfrac{1}{2}\log\frac{n^2+|k|^2}{|k|^2} - \frac{1}{2}\frac{n}{n^2+|k|^2} - \\ &\quad - \frac{1}{12}\left[\frac{n^2-|k|^2}{(n^2+|k|^2)^2} - \frac{1}{10}\frac{n^4-6n^2|k|^2+|k|^4}{(n^2+|k|^2)^4} + \ldots\right]\end{aligned}$$

(the coefficients of the terms in the numerators of the fractions inside the square bracket are alternate binomial coefficients). With $n = 10$ and $\gamma = 1/k^2$, this gives

$$\begin{aligned}\operatorname{Re}\psi(10+i|k|) - \log k &= \tfrac{1}{2}\log(1+100|\gamma|) - \frac{1}{2}\frac{10|\gamma|}{1+100|\gamma|} - \\ &\quad - \frac{1}{12}\left[|\gamma|\frac{100|\gamma|-1}{(100|\gamma|+1)^2} - \frac{1}{10}|\gamma|^2\frac{(100|\gamma|)^2-6(100|\gamma|)+1}{(100|\gamma|+1)^4} + \right.\\ &\quad \left. + \frac{1}{21}|\gamma|^3\frac{(100|\gamma|)^3-15(100|\gamma|)^2+15(100|\gamma|)-1}{(100|\gamma|+1)^6}\ldots\right]\end{aligned} \tag{11}$$

and the real part of (9) gives

$$\mathrm{Re}[\psi(k)-\log k+1/2k]$$
$$= \mathrm{Re}\,\psi(10+i|k|)-\log|k|-\left[\frac{1}{1^2+|k|^2}+\frac{2}{2^2+|k|^2}+\frac{3}{3^2+|k|^2}+\ldots+\frac{9}{9^2+|k|^2}\right]$$
$$= \mathrm{Re}\,\psi(10+i|k|)-\log|k|-|\gamma|\left[\frac{1}{1^2|\gamma|+1}+\frac{2}{2^2|\gamma|+1}+\ldots+\frac{9}{9^2|\gamma|+1}\right]. \quad (12)$$

Since the calculations were done for exact values of γ, the forms (11) and (12) were very convenient for computation.

4. Results

Values of $\psi(k)-\log k+1/2k$ were evaluated to 10 decimals for $\gamma = 1/k^2 = -1{\cdot}00(0{\cdot}02)+1{\cdot}00$, and differenced. The differences indicated that random errors due to rounding-off were not more than 2 in the tenth decimal, although, for negative values of γ, each value was calculated as the sum of about 12 terms. The values were subtabulated to 0·01 intervals in γ and rounded-off to 8 decimals; the results were differenced on a 'National' machine and are given in the table. The maximum error in a tabulated value should not be more than 0·6 in the eighth decimal.

The first and second differences of the function values are given in the table; it is never necessary to take fourth differences into account, and linear interpolation is adequate to give six-decimal accuracy.

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TABLE

$\gamma = 1/k^2$	$\psi(k) - \log k + 1/2k$	δ^1	δ^2	$\gamma = 1/k^2$	$\psi(k) - \log k + 1/2k$	δ^1	δ^2
−1·00	+0·09465032	−102865	+ 35	−0·50	+0·04451209	−95632	+257
0·99	·09362167	102826	39	·49	·04355577	95370	262
·98	·09259341	102784	42	·48	·04260207	95106	264
·97	·09156557	102737	47	·47	·04165101	94836	270
·96	·09053820	102688	49	·46	·04070265	94565	271
−0·95	+0·08951132	−102634	+ 54	−0·45	+0·03975700	−94289	+276
·94	·08848498	102577	57	·44	·03881411	94011	278
·93	·08745921	102515	62	·43	·03787400	93730	281
·92	·08643406	102451	64	·42	·03693670	93447	283
·91	·08540955	102382	69	·41	·03600223	93162	285
−0·90	+0·08438573	−102308	+ 74	−0·40	+0·03507061	−92874	+288
·89	·08336265	102231	77	·39	·03414187	92585	289
·88	·08234034	102150	81	·38	·03321602	92294	291
·87	·08131884	102064	86	·37	·03229308	92003	291
·86	·08029820	101975	89	·36	·03137305	91711	292
−0·85	+0·07927845	−101881	+ 94	−0·35	+0·03045594	−91418	+293
·84	·07825964	101782	99	·34	·02954176	91126	292
·83	·07724182	101679	103	·33	·02863050	90834	292
·82	·07622503	101572	107	·32	·02772216	90543	291
·81	·07520931	101461	111	·31	·02681673	90253	290
−0·80	+0·07419470	−101344	+117	−0·30	+0·02591420	−89965	+288
·79	·07318126	101223	121	·29	·02501455	89678	287
·78	·07216903	101098	125	·28	·02411777	89395	283
·77	·07115805	100967	131	·27	·02322382	89113	282
·76	·07014838	100833	134	·26	·02233269	88835	278
−0·75	+0·06914005	−100693	+140	−0·25	+0·02144434	−88562	+273
·74	·06813312	100549	144	·24	·02055872	88291	271
·73	·06712763	100399	150	·23	·01967581	88025	266
·72	·06612364	100245	154	·22	·01879556	87764	261
·71	·06512119	100087	158	·21	·01791792	87508	256
−0·70	+0·06412032	− 99922	+165	−0·20	+0·01704284	−87256	+252
·69	·06312110	99754	168	·19	·01617028	87011	245
·68	·06212356	99580	174	·18	·01530017	86771	240
·67	·06112776	99401	179	·17	·01443246	86535	236
·66	·06013375	99218	183	·16	·01356711	86307	228
−0·65	+0·05914157	− 99030	+188	−0·15	+0·01270404	−86083	+224
·64	·05815127	98836	194	·14	·01184321	85865	218
·63	·05716291	98637	199	·13	·01098456	85653	212
·62	·05617654	98435	202	·12	·01012803	85445	208
·61	·05519219	98226	209	·11	·00927358	85241	204
−0·60	+0·05420993	− 98014	+212	−0·10	+0·00842117	−85044	+197
·59	·05322979	97796	218	·09	·00757073	84849	195
·58	·05225183	97573	223	·08	·00672224	84660	189
·57	·05127610	97347	226	·07	·00587564	84472	188
·56	·05030263	97114	233	·06	·00503092	84290	182
−0·55	+0·04933149	− 96878	+236	−0·05	+0·00418802	−84109	+181
·54	·04836271	96638	240	·04	·00334693	83932	177
·53	·04739633	96392	246	·03	·00250761	83758	174
·52	·04643241	96143	249	·02	·00167003	83586	172
·51	·04547098	95889	254	·01	·00083417	83417	169
−0·50	+0·04451209		+257	−0·00	+0·00000000		+167

TABLE (*cont.*)

$\gamma = 1/k^2$	$\psi(k) - \log k + 1/2k$	δ^1	δ^2	$\gamma = 1/k^2$	$\psi(k) - \log k + 1/2k$	δ^1	δ^2
+0·00	−0·00000000	−83250	+167	+0·50	−0·03992983	−76842	+104
·01	·00083250	83086	164	·51	·04069825	76740	102
·02	·00166336	82924	162	·52	·04146565	76639	101
·03	·00249260	82764	160	·53	·04223204	76538	101
·04	·00332024	82607	157	·54	·04299742	76437	101
+0·05	−0·00414631	−82450	+157	+0·55	−0·04376179	−76337	+100
·06	·00497081	82296	154	·56	·04452516	76239	98
·07	·00579377	82145	151	·57	·04528755	76140	99
·08	·00661522	81994	151	·58	·04604895	76043	97
·09	·00743516	81845	149	·59	·04680938	75946	97
+0·10	−0·00825361	−81699	+146	+0·60	−0·04756884	−75849	+ 97
·11	·00907060	81554	145	·61	·04832733	75753	96
·12	·00988614	81410	144	·62	·04908486	75659	94
·13	·01070024	81268	142	·63	·04984145	75563	96
·14	·01151292	81128	140	·64	·05059708	75470	93
+0·15	−0·01232420	−80989	+139	+0·65	−0·05135178	−75377	+ 93
·16	·01313409	80852	137	·66	·05210555	75284	93
·17	·01394261	80715	137	·67	·05285839	75191	93
·18	·01474976	80580	135	·68	·05361030	75100	91
·19	·01555556	80448	132	·69	·05436130	75009	91
+0·20	−0·01636004	−80315	+133	+0·70	−0·05511139	−74918	+ 91
·21	·01716319	80185	130	·71	·05586057	74829	89
·22	·01796504	80055	130	·72	·05660886	74739	90
·23	·01876559	79926	129	·73	·05735625	74650	89
·24	·01956485	79800	126	·74	·05810275	74562	88
+0·25	−0·02036285	−79673	+127	+0·75	−0·05884837	−74473	+ 89
·26	·02115958	79549	124	·76	·05959310	74387	86
·27	·02195507	79425	124	·77	·06033697	74299	88
·28	·02274932	79302	123	·78	·06107996	74214	85
·29	·02354234	79181	121	·79	·06182210	74127	87
+0·30	−0·02433415	−79060	+121	+0·80	−0·06256337	−74042	+ 85
·31	·02512475	78941	119	·81	·06330379	73957	85
·32	·02591416	78822	119	·82	·06404336	73873	84
·33	·02670238	78705	117	·83	·06478209	73789	84
·34	·02748943	78588	117	·84	·06551998	73705	84
+0·35	−0·02827531	−78473	+115	+0·85	−0·06625703	−73623	+ 82
·36	·02906004	78358	115	·86	·06699326	73540	83
·37	·02984362	78245	113	·87	·06772866	73457	83
·38	·03062607	78131	114	·88	·06846323	73376	81
·39	·03140738	78020	111	·89	·06919699	73295	81
+0·40	−0·03218758	−77908	+112	+0·90	−0·06992994	−73214	+ 81
·41	·03296666	77799	109	·91	·07066208	73133	81
·42	·03374465	77689	110	·92	·07139341	73054	79
·43	·03452154	77580	109	·93	·07212395	72974	80
·44	·03529734	77473	107	·94	·07285369	72895	79
+0·45	−0·03607207	−77366	+107	+0·95	−0·07358264	−72816	+ 79
·46	·03684573	77260	106	·96	·07431080	72738	78
·47	·03761833	77154	106	·97	·07503818	72660	78
·48	·03838987	77050	104	·98	·07576478	72583	77
·49	·03916037	76946	104	·99	·07649061	72505	78
+0·50	−0·03992983		+104	+1·00	−0·07721566		+ 77

FINITE DIFFERENCE FORMULAE FOR THE SQUARE LATTICE

By W. G. BICKLEY (*Imperial College, London*)

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SUMMARY

The paper gives approximate formulae for derivatives (including combinations like ∇^2 and ∇^4), and integrals, of a function of two independent variables, in terms of its values at nodes of a square lattice, primarily for use in the numerical solution of partial differential equations. Consideration is given to the *form*, as well as to the magnitude, of the leading terms in the error, and what is believed to be for most purposes optimum combinations are thus selected for the simpler compact sets of nodes.

1. Introduction

THE use of numerical methods in solving problems in applied mathematics is becoming increasingly common for partial, as well as for ordinary, differential equations. The first step in such a process is usually the replacement of differential equations by their finite difference approximations. For ordinary differential equations, approximations of various orders of accuracy to derivatives and integrals are well known and listed,† but for partial differential equations it is as yet rare to find any but the crudest approximations employed.‡§

For many purposes formulae of the Lagrangian type are convenient. In dealing with partial differential equations with two independent variables, equidistant, and equal, intervals in the independent variables will usually be employed, so that the required function is computed at the nodes of a square lattice (net). The object of this paper is to explore the possibilities of refining approximations to the quantities—derivatives and integrals—which occur in the formulation of problems in applied mathematics.

Considerable use is made of symbolic operators, which are powerful in developing such formulae; *a posteriori* proofs of the formulae are readily constructed—if they are deemed necessary.

† W. G. Bickley, 'Formulae for numerical integration', *Math. Gaz.* **23** (1939), 352–9; id., 'Formulae for numerical differentiation', ibid. **25** (1941), 19–26; L. J. Comrie, *Interpolation and Allied Tables*, H.M. Stationery Office (reprinted 1942).

‡ Some attempt is made by A. Thom, 'Arithmetical solution of equations of the type $\nabla^4\psi$ = const.', *A.R.C., R. and M.* (1933), 1604.

§ Formulae of various types are given by L. Collatz, *Eigenwertproblem und ihre numerische Behandlung*, Akad. Verlag. (Leipzig, 1945).

We refrain from giving examples of the use of the formulae since this is quite straightforward. Nor do we advocate that the refinements should be used in the early stages, or on the coarser nets. Their proper place is in the *final* stages, where they may add considerably to the accuracy of a solution with but little extra labour and without further reducing the mesh length.

2. Symmetric sums

We cover the (x, y)-plane with a square lattice of mesh length a, and label a typical point 0 (see Fig. 1). Neighbouring points are numbered

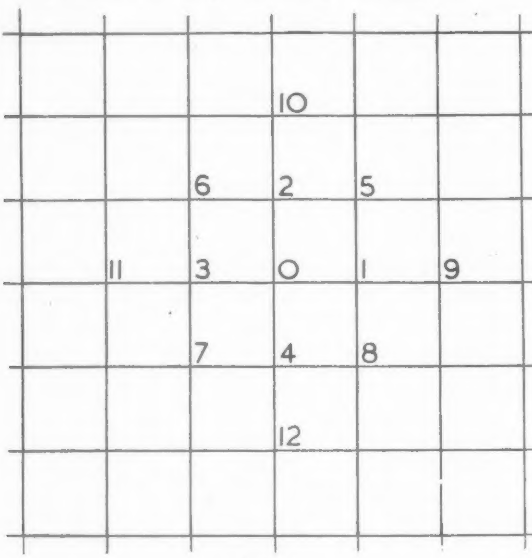

FIG. 1.

1, 2,..., 12, as in the figure, and these numbers will be used as subscripts to indicate that the value of any quantity (usually $f(x, y)$) is to be taken at the point in question.

We commence by recalling the symbolic form of Taylor's series in one variable,

$$f(x+h) = e^{hD}f(x), \tag{1}$$

where $$D \equiv d/dx.$$

To deal with two independent variables (x, y) with a mesh length a it is convenient to introduce the symbolic operators

$$\xi \equiv a\,\partial/\partial x, \qquad \eta \equiv a\,\partial/\partial y. \tag{2}$$

In terms of these we may write

$$\left.\begin{aligned} f_1 &= e^{\xi}f_0, & f_2 &= e^{\eta}f_0 \\ f_3 &= e^{-\xi}f_0, & f_4 &= e^{-\eta}f_0 \\ f_5 &= e^{\xi+\eta}f_0, & & \end{aligned}\right\} \tag{3}$$

and so on.

We note that
$$\xi^2+\eta^2 = a^2\nabla^2 \tag{4}$$
and find it convenient to write
$$\xi\eta = a^2\frac{\partial^2}{\partial x\partial y} = a^2\mathscr{D}^2. \tag{5}$$

Now it will usually happen that the quantities occurring in the mathematical equations will be independent of the choice of axes—invariant for rotation of these axes—and clearly values of such quantities at 0 can be represented or approximated only by *sums* of values at points *symmetrically disposed* about 0. Of such 'symmetric sums' three only (apart from f_0 itself) are simple (in the sense that they include only four terms), and involve points not too remote from 0, namely,

$$\begin{aligned} S_1 = f_1+f_2+f_3+f_4 &= (e^{\xi}+e^{\eta}+e^{-\xi}+e^{-\eta})f_0 = 2(\cosh\xi+\cosh\eta)f_0 \\ &= 4f_0+(\xi^2+\eta^2)f_0+\frac{2}{4!}(\xi^4+\eta^4)f_0+\frac{2}{6!}(\xi^6+\eta^6)f_0+\dots \\ &= 4f_0+a^2\nabla^2 f_0+\tfrac{1}{12}a^4(\nabla^4-2\mathscr{D}^4)f_0+\tfrac{1}{360}a^6(\nabla^6-3\nabla^2\mathscr{D}^4)f_0\dots. \end{aligned} \tag{6}$$

$$\begin{aligned} S_2 &= f_5+f_6+f_7+f_8 \\ &= (e^{\xi+\eta}+e^{-\xi+\eta}+e^{-\xi-\eta}+e^{\xi-\eta})f_0 \\ &= 4\cosh\xi\cosh\eta f_0 \\ &= 4f_0+\frac{4}{2!}(\xi^2+\eta^2)f_0+\frac{4}{4!}(\xi^4+6\xi^2\eta^2+\eta^4)f_0+ \\ &\qquad\qquad +\frac{4}{6!}(\xi^6+15\xi^4\eta^2+15\xi^2\eta^4+\eta^6)f_0+\dots \\ &= 4f_0+2a^2\nabla^2 f_0+\tfrac{1}{6}a^4(\nabla^4+4\mathscr{D}^4)f_0+\tfrac{1}{180}a^6(\nabla^6+12\nabla^2\mathscr{D}^4)f_0+\dots. \end{aligned} \tag{7}$$

$$\begin{aligned} S_3 &= f_9+f_{10}+f_{11}+f_{12} = (e^{2\xi}+e^{2\eta}+e^{-2\xi}+e^{-2\eta})f_0 \\ &= 4f_0+4a^2\nabla^2 f_0+\tfrac{4}{3}a^4(\nabla^4-2\mathscr{D}^4)f_0+\tfrac{8}{45}(\nabla^6-3\nabla^2\mathscr{D}^4)f_0+\dots. \end{aligned} \tag{8}$$

The next symmetrical sum involves eight terms:

$$f(\pm 2a, \pm a) \quad \text{and} \quad f(\pm a, \pm 2a).$$

3. Approximations to $\nabla^2 f$

By equation (6) we have immediately the well-known and most frequently applied approximation to $\nabla^2 f$, namely,

$$\nabla^2 f_0 = \{S_1-4f_0\}/a^2+O(a^2), \tag{9}$$

usually attributed to Liebmann.

Equations (7) and (8) lead to what is essentially the same result for mesh lengths $a\sqrt{2}$ and $2a$ respectively.

The problem of improving the accuracy *conveniently* is not simple. Clearly the use of S_1 and S_2 cannot eliminate *both* ∇^4 and $\mathscr{D}^4$, and although we can eliminate both by using S_1 and S_3, we use points distant $2a$ from 0, which complicates procedure at a boundary, and also we cannot then obtain any help from S_2.

The use we can best make of S_1 and S_2 is to eliminate the term in $\mathscr{D}^4$, since this is not invariant for rotation of axes. The resulting 'error' term in a^4 then involves only $\nabla^4 f_0$, and is thus (*a*) invariant, (*b*) calculable (at least approximately) from the values of $\nabla^2 f$, and (*c*) small (theoretically zero) for Laplace's equation.

The resulting formula is

$$4S_1+S_2 = 20f_0+6a^2\nabla^2 f_0+\tfrac{1}{2}a^4\nabla^4 f_0+\tfrac{1}{60}a^6(\nabla^4+2\mathscr{D}^4)\nabla^2 f_0+\ldots, \tag{10}$$

and from this†

$$\nabla^2 f_0 = \{4S_1+S_2-20f_0\}/6a^2-\tfrac{1}{12}a^2\nabla^4 f_0+O(a^4). \tag{11}$$

One may use (9) to compute $\nabla^4 f_0$ from the (approximate) values of $\nabla^2 f$, with an error of order a^2, so that if the term $-\frac{1}{12}a^2\nabla^4 f_0$ is allowed for in this way, the error in (11) is of order a^4. This procedure is equivalent to the use of a fourth difference correction (as advocated by Fox),‡ and has, indeed, the advantage that it is a *two*-dimensional correction, whereas all Fox's fourth (and higher) differences are differences of one-dimensional sequences.

We can, alternatively, eliminate completely the a^4 terms between (6) and (8), obtaining

$$16S_1-S_3 = 60f_0+12a^2\nabla^2 f_0-\tfrac{2}{15}a^6(\nabla^4-3\mathscr{D}^4)\nabla^2 f_0\ldots \tag{12}$$

or

$$\nabla^2 f_0 = \{16S_1-S_3-60f_0\}/12a^2+O(a^4). \tag{13}$$

If f_0 is an approximate solution of Laplace's equation, then the last term written in (12) shows that the 'error' of (13) is of order higher than a^4. But, as already indicated, the superior accuracy of (13) is purchased by wider spread of the points used.

Using S_1, S_2, and S_3 we may eliminate $\nabla^2 f_0$ and $\mathscr{D}^4 f_0$, and obtain the well-known approximation,

$$\nabla^4 f_0 = \{20f_0-8S_1+2S_2+S_3\}/a^4+O(a^2), \tag{14}$$

and clearly nothing better is available unless we include additional points.

† This formula is given, effectively, by P. M. and A. M. Woodward, 'Four-figure tables of the Airy function in the complex plane', *Phil. Mag.* (7) **37** (1946), 259.

‡ L. Fox, in his paper 'Some improvements in the use of relaxation methods for the solution of ordinary and partial differential equations', *Proc. Roy. Soc.* A, **190** (1947), 31–59, corrects the crude finite difference approximation by the use of higher differences.

4. Other derivatives

First-order derivatives occur infrequently in the governing differential equations, but are common in boundary conditions.

The crudest approximation to $(\partial f/\partial x)_0$ is Euler's, namely, $(f_1-f_0)/a$, but this is an approximation 'centred' at the mid-point of the mesh 01. For the value centred at 0 we use

$$\begin{aligned} f_1-f_3 &= 2\sinh\xi f_0 \\ &= 2\left\{\xi+\frac{1}{3!}\xi^3+\frac{1}{5!}\xi^5+\ldots\right\}f_0, \end{aligned} \tag{15}$$

so that
$$(\partial f/\partial x)_0 = (f_1-f_3)/2a+O(a^2). \tag{16}$$

To improve this we must clearly use approximations to $(\partial f/\partial x)_2$ and $(\partial f/\partial x)_4$, in equal proportions. Now

$$\begin{aligned} f_5-f_6+f_8-f_7 &= (e^{\xi+\eta}-e^{-\xi+\eta}+e^{\xi-\eta}-e^{-\xi-\eta})f_0 \\ &= 4\sinh\xi\cosh\eta f_0 \\ &= 4\left\{1+\frac{1}{3!}\xi^2+\frac{1}{2!}\eta^2+\ldots\right\}\xi f_0. \end{aligned} \tag{17}$$

It is clear that potentially the most useful combination will be one in which (since they cannot be eliminated) the third-order terms constitute a multiple of $\nabla^2(\partial f/\partial x)$. We readily find the combination

$$\begin{aligned} 4(f_1-f_3)+(f_5-f_6)+(f_8-f_7) &= \{12+2(\xi^2+\eta^2)+\ldots\}\xi f_0 \\ &= 12a(\partial f/\partial x)_0+2a^3\nabla^2(\partial f/\partial x)_0+O(a^5).\ldots \end{aligned} \tag{18}$$

By using the approximation similar to (9) in the second term on the right-hand side of (18), the relative error in $(\partial f/\partial x)_0$ may be made of order a^4.

When the point 0 is a boundary point, the use of the above formulae necessitates values at 'fictitious' points, outside the boundary, but the use of such points is a commonplace method of the 'relaxation' and other techniques.

Boundary derivatives at points other than nodes, and normal or tangential derivatives for curved boundaries, do not lend themselves to any simple treatment if any improvement upon the crudest approximation is desired.

The formula for $(\partial f/\partial y)_0$, corresponding to (18), may immediately be written down.

5. Surface integrals

In many problems the integral of a function taken over some region, determined at the nodes of a lattice, is required.

The simplest approximation (corresponding to the trapezoidal rule in one dimension) associates the value of the integrand at the node with the square of side a centred at the node (Fig. 2), and applies a factor $\frac{1}{2}$ to nodes on a boundary and $\frac{1}{4}$ to nodes at corners.

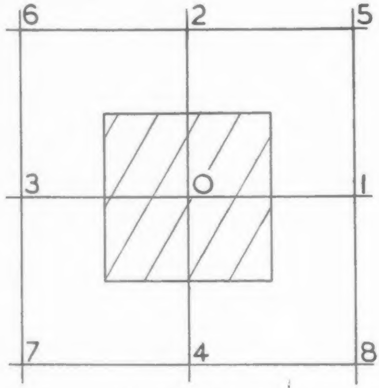

FIG. 2.

An obvious improvement is to use Simpson's rule in both directions. According to this

$$\int_{-a}^{a}\int_{-a}^{a} f\,dxdy \doteqdot \frac{a^2}{9}(16f_0+4S_1+S_2), \tag{19}$$

but this is not the best that can be done with the nine nodes 0, 1,..., 8.

We have

$$\int_{-a}^{a}\int_{-a}^{a} f(x,y)\,dxdy = a^2\int_{-1}^{1}\int_{-1}^{1} e^{r\xi+s\eta}f_0\,drds$$

$$= 4a^2\frac{\sinh\xi\sinh\eta}{\xi\eta}f_0$$

$$= 4a^2\left\{1+\frac{1}{3!}(\xi^2+\eta^2)+\frac{1}{5!}(\xi^4+\tfrac{10}{3}\xi^2\eta^2+\eta^4)+\dots\right\}f_0$$

$$= 4a^2\{1+\tfrac{1}{6}a^2\nabla^2+\tfrac{1}{120}a^4(\nabla^4+\tfrac{4}{3}\mathscr{D}^4)+\dots\}f_0. \tag{20}$$

By the use of f_0, S_1, and S_2 we can obtain the correct amounts of any three of f_0, $\nabla^2 f_0$, $\nabla^4 f_0$, and $\mathscr{D}^4 f_0$—but *not* of all four. If one must remain, clearly it should best be $\nabla^4 f_0$. We find, with no difficulty,

$$\int_{-a}^{a}\int_{-a}^{a} f(x,y)\,dxdy = \frac{a^2}{45}(88f_0+16S_1+7S_2)-\tfrac{1}{45}a^6\nabla^4 f_0+O(a^8). \tag{21}$$

The numerical coefficients have become larger—beyond the scope of mental arithmetic—but the formula has the merit that the value of the leading term of the error is readily calculable, and is small for solutions of the Laplace equation.

We may also compare (19) with (20), and the result is

$$\int_{-a}^{a}\int_{-a}^{a} f(x,y)\,dxdy = \frac{a^2}{9}(16f_0+4S_1+S_2)-\frac{a^6}{45}(\nabla^4-2\mathscr{D}^4)f_0+O(a^8), \tag{22}$$

in which the term in a^6 is not readily computed.

Two cruder formulae may be mentioned, giving approximations to the same double integral, namely

$$\tfrac{2}{3}a^2(2f_0+S_1)+O(a^6), \tag{23}$$

$$\tfrac{1}{3}a^2(8f_0+S_2)+O(a^6). \tag{24}$$

Finally, it may be worth recording the approximation to the integral over the square of side a centred at the point 0 (Fig. 2), in terms of the values at the points 0, 1,..., 8.

$$\begin{aligned}\int_{-\frac{1}{2}a}^{\frac{1}{2}a}\int_{-\frac{1}{2}a}^{\frac{1}{2}a} f(x,y)\,dxdy &= 4a^2\frac{\sinh\frac{1}{2}\xi\sinh\frac{1}{2}\eta}{\xi\eta}f_0\\ &= a^2\{1+\tfrac{1}{24}a^2\nabla^2+\tfrac{1}{1920}a^4(\nabla^4+\tfrac{4}{3}\mathscr{D}^4)+...\}f_0,\end{aligned} \tag{25}$$

and comparison with (6) and (7) leads to the result

$$\int_{-\frac{1}{2}a}^{\frac{1}{2}a}\int_{-\frac{1}{2}a}^{\frac{1}{2}a} f(x,y)\,dxdy = \frac{a^2}{1440}(1244f_0+38S_1+11S_2)-\tfrac{17}{5760}a^6\nabla^4 f_0+O(a^8). \tag{26}$$

It is at first sight attractive to apply this successively to all nodes of a region. The result is that the factor for all internal nodes is unity—which clearly cannot be expected to give the best results. The explanation lies in the fact that the regions to be associated with nodes on the boundary are *not complete* squares but half-squares or (at corners) quarter-squares. The contributions from these are *not* given by symmetric sums, so that the corrections must be applied to a border of nodes on and within the boundary. The situation is parallel to the Gregory integration formula in one dimension. Corresponding correction terms could be worked out, but their use would be a complication which (21) or (22) completely avoids.

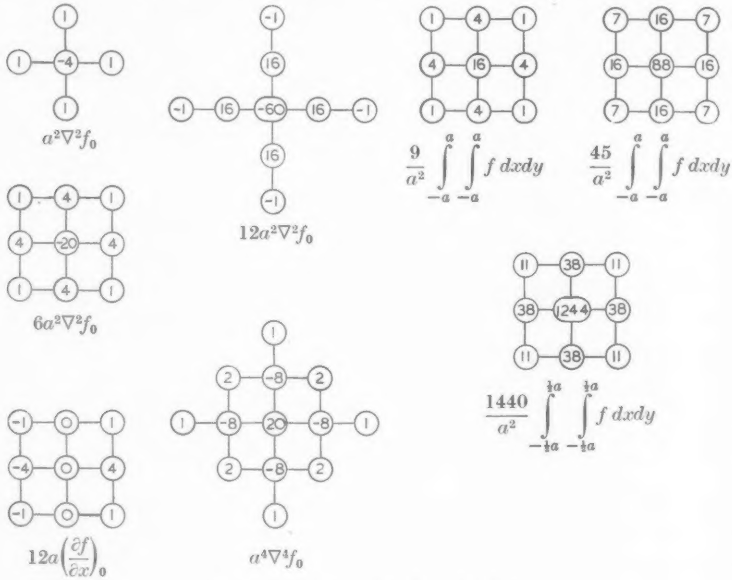

FIG. 3. Computational molecules.

6. Computational molecules

It seems serviceable to represent pictorially the formula given above, in the form of diagrams indicating the factor to be applied to the function value at the corresponding node. In each case the result of adding the given multiples of the function at the corresponding points is given, but for the 'error' terms reference must be made to the appropriate equation.

ON LAMINAR BOUNDARY-LAYER FLOW NEAR A POSITION OF SEPARATION

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SUMMARY

Singularities are considered in the solution of the laminar boundary-layer equations at a position of separation. A singularity of the type here considered occurred in a careful numerical computation by Hartree for a linearly decreasing velocity distribution outside the boundary layer; it may occur generally. Whenever it does occur, the boundary-layer equations cease to be valid at and near separation on the upstream side, and also downstream of separation. The work suggests that singularities may arise in the solution of non-linear parabolic equations due to their non-linearity. The formulae found may help computers of laminar boundary layers, who desire more than a rough solution, to have an end-point at which to aim.

1. Introduction and summary

For a flow at a large Reynolds number along an immersed solid surface a boundary layer is formed through which the velocity rises rapidly from zero at the surface to its value in the main stream. The approximate equations for the two-dimensional flow of a fluid of constant density ρ and kinematic viscosity ν in a boundary layer are

$$\left.\begin{aligned} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2} \\ u=\frac{\partial\psi}{\partial y}, &\qquad v=-\frac{\partial\psi}{\partial x} \end{aligned}\right\}, \tag{1}$$

where x is distance measured along the solid boundary in the plane of the flow, y is distance normal to the surface, u, v are the velocity components in the directions of x and y increasing, p the pressure, and ψ the stream function. According to the approximations of boundary-layer theory, p and $\partial p/\partial x$ may be taken independent of y, and if U is the velocity just outside the boundary layer in the main stream,

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}=U\frac{dU}{dx}. \tag{2}$$

Moreover, according to these approximations v/u and $(\partial u/\partial x)/(\partial u/\partial y)$ are small.

As boundary conditions we have that $u=0$ and $v=0$ (or $\psi=0$) at $y=0$, u is given as a function of y for some initial value of x, and the

velocity passes over smoothly into the velocity of the main stream, i.e. $u \to U$, $\partial u/\partial y \to 0$, $\partial^2 u/\partial y^2 \to 0$, etc., as $y \to \infty$.

If $dU/dx < 0$ and $\partial p/\partial x > 0$, then $\partial u/\partial y$ at the wall $y = 0$ decreases as x increases until it vanishes; beyond the section at which it is zero, a slow back flow sets in along the wall, and the boundary layer separates from the surface.

Let d be any representative length of the system, U_0 a representative velocity, such as the undisturbed stream velocity, and R the Reynolds number $U_0 d/\nu$. The equations may be made non-dimensional by writing

$$x' = x/d, \quad y' = R^{\frac{1}{2}}y/d, \quad u' = u/U_0, \quad v' = R^{\frac{1}{2}}v/U_0, \quad p' = p/\rho U_0^2. \tag{3}$$

In the non-dimensional form, let $x' = 0$ be the initial section, at which u' is given as or approximated by a polynomial or power-series

$$u' = a_1 y' + a_2 y'^2 + \ldots, \tag{4}$$

and let

$$-\frac{\partial p'}{\partial x'} = p_0 + p_1 x' + p_2 x'^2 + \ldots. \tag{5}$$

Then it is known that if there is to be a solution without singularities, certain equations must be satisfied:

$$2a_2 + p_0 = 0, \quad a_3 = 0, \quad 5!\,a_5 + 2a_1 p_1 = 0, \quad 6!\,a_6 - 2p_0 p_1 = 0, \text{ etc.} \tag{6}$$

Only a_1, a_4, a_7,... are at our disposal. When the conditions are broken, the solution has an algebraic singularity at $x = 0$ (**1**).†

At the position of separation $\partial u/\partial y = 0$ at $y = 0$, i.e. $a_1 = 0$. The conditions for the absence of singularities when $a_1 = 0$ are considerably more complicated than those above.‡ If we suppose u expanded in a power series in x (we drop the dashes for the present)

$$u = u_0 + u_1 x + u_2 x^2 + \ldots, \tag{7}$$

where u_0, u_1, u_2 are functions of y, expressible as power series,

$$\left.\begin{aligned} u_0 &= a_2 y^2 + a_3 y^3 + \ldots \\ u_1 &= b_1 y + b_2 y^2 + \ldots \\ u_2 &= c_1 y + c_2 y^2 + \ldots \end{aligned}\right\}, \tag{8}$$

the conditions are

$$\left.\begin{aligned} 2a_2 + p_0 = 0, \quad a_3 = 0, \quad a_4 = 0, \quad a_5 = 0, \quad 6!\,a_6 = 2p_0 p_1, \\ a_7 = 0, \quad a_9 = 0, \text{ etc.} \end{aligned}\right\}. \tag{9}$$

Only a_8, a_{12}, a_{16}, a_{20},... are at our disposal. In addition b_1, c_1, d_1,... are determined, not from the equations for u_1, u_2, u_3,... respectively, but from

† In the last term in equation (5) on p. 4 of ref. (**1**), the denominator should be $8a_1^{\frac{3}{2}}$, not $4a_1^{\frac{3}{2}}$; I am indebted to Prof. Hartree for this correction. See also ref. (**2**) at end.

‡ Goldstein, loc. cit., pp. 17, 18. [In the last line of p. 17 of ref. (**1**), in the equations of the footnote, for $6!\,a_6 = 2p_0 p_2$, read $6!\,a_6 = 2p_0 p_1$.]

the conditions for the absence of singularities in u_2, u_3, u_4,... respectively. There is also an ambiguity of sign, which can be determined only from physical considerations. If the conditions are broken there is a formal solution for the flow downstream of the form

$$\left.\begin{aligned} \psi &= \xi^3[f_0(\eta)+\xi f_1(\eta)+...] \\ u &= \tfrac{1}{4}\xi^2[f_0'(\eta)+\xi f_1'(\eta)+...] \end{aligned}\right\}, \tag{10}$$

where
$$\xi = x^{\frac{1}{4}}, \qquad \eta = y/4x^{\frac{1}{4}}. \tag{11}$$

This formal solution fails, however, in certain circumstances, one of which is that the condition

$$2a_2+p_0 = 0 \tag{12}$$

is satisfied, while the other conditions are not satisfied.

No other work has been reported on possible singularities at separation.

No analytical solution is known for a boundary-layer flow involving separation, and the methods used are approximate and numerical. The published methods of computation are rather rough, but recently more exact methods have been suggested and tried. The work described here arose out of an unpublished communication from Professor Hartree, in which he repeated Dr. Howarth's computation (**3**) for a linearly decreasing velocity distribution, $U = \beta_0 - \beta_1 x$, with $u = U$ at $x = 0$. Professor Hartree replaces the partial derivatives with respect to x by finite differences, and retains the y-derivatives, so the partial differential equation is replaced approximately by a sequence of ordinary differential equations, each of which relates the velocity distribution through the boundary layer at one section to that at another section a short distance upstream, where it is known. The ordinary differential equations were solved laboriously on hand calculating machines rather than on the Differential Analyser in order that more significant figures might be retained.

Now all computations in which any attempt was made to obtain real accuracy at and near separation seem to have met with considerable difficulty. As a result of his computations, Professor Hartree was convinced that there was a singularity in the solution at the position of separation, and I undertook to try to find some formulae that would hold near this singularity and would help in finishing the computation.

To study the singularity near separation, the equations are put into non-dimensional form in a special way. Let x_s, U_s, U_s' be the values of x, U, dU/dx at separation, so that $U_s > 0$, $U_s' < 0$. We are not interested in any other properties or dimensions of the system, so as representative length l and Reynolds number R we take

$$l = -U_s/U_s', \qquad R = U_s l/\nu. \tag{13}$$

We are also concerned with the flow upstream of separation, so for our non-dimensional distances we write

$$x_1 = (x_s - x)/l, \qquad y_1 = R^{\frac{1}{2}}y/l. \tag{14}$$

Also put

$$u_1 = u/U_s, \quad v_1 = R^{\frac{1}{2}}v/U_s, \quad U_1 = U/U_s, \quad p_1 = p/\rho U_s^2, \quad \psi_1 = R^{\frac{1}{2}}\psi/lU_s. \tag{15}$$

The equations become

$$\left.\begin{aligned} -u_1\frac{\partial u_1}{\partial x_1}+v_1\frac{\partial u_1}{\partial y_1} &= \frac{\partial p_1}{\partial x_1}+\frac{\partial^2 u_1}{\partial y_1^2} \\ u_1 = \frac{\partial \psi_1}{\partial y_1}, \qquad v_1 &= \frac{\partial \psi_1}{\partial x_1} \\ \frac{\partial p_1}{\partial x_1} &= -U_1\frac{dU_1}{dx_1} \end{aligned}\right\}. \tag{16}$$

It is easy to see that $U_1 = 1$ and $dU_1/dx_1 = 1$ at $x_1 = 0$, so we may write

$$\frac{\partial p_1}{\partial x_1} = -(1+P_1x_1+P_2x_1^2+...), \tag{17}$$

i.e. p_0 (in our previous notation) $= -1$. For the linear velocity distribution $U = \beta_0 - \beta_1 x$,

$$\frac{\partial p_1}{\partial x_1} = -(1+x_1), \qquad P_1 = 1, \qquad P_2 = P_3 = ... = 0. \tag{18}$$

Formulae for the P_1, P_2,... are easily found in the general case, and if U is taken as a given function of x, it is easily found that the P's are independent of the position of separation if

$$U = (\text{constant})e^{-\beta x}, \quad \text{or} \quad U = (\beta_0-\beta_1 x)^m \quad \text{or} \quad (\beta_0+\beta_1 x)^{-m}, \tag{19}$$

the constants β, β_0, β_1, m being positive. (However, since $l = -U_s/U_s'$, the scale varies as x_s varies in the last two cases.) For other values of U, the P's depend on x_s, the position of separation.

The boundary conditions are $\psi_1 = 0$, $u_1 = 0$ at $y_1 = 0$, and $u_1 \to U_1$, etc., as $y_1 \to \infty$. Since $x_1 = 0$ is a position of separation, $(\partial u_1/\partial y_1)_{y_1=0} = 0$ at $x_1 = 0$, so

$$u_1 = a_2 y_1^2 + a_3 y_1^3 + ... \quad \text{at} \quad x_1 = 0. \tag{20}$$

Singularities in the solution for the corresponding system of equations for the motion downstream have been considered (see equations (10) and (11)); near $x_1 = 0$, $y_1 = 0$, ψ_1 is a function of $x_1^{\frac{3}{4}}$ and $y_1/x_1^{\frac{1}{4}}$. The skin-friction is $\mu(\partial u/\partial y)_{y=0}$ and the determining quantity is $(\partial u_1/\partial y_1)_{y_1=0}$, which is an ascending series of powers of $x_1^{\frac{1}{4}}$, beginning with a multiple of $x_1^{\frac{1}{2}}$. If, however, $2a_2+p_0 = 0$, which corresponds to $a_2 = \frac{1}{2}$ in the new notation, there are special features in the solution; in particular the series for $(\partial u_1/\partial y_1)_{y_1=0}$ now begins with a term in $x_1^{\frac{1}{4}}$. Now Professor Hartree was

quite certain that this particular feature was present in his computed solution. The calculations reported here therefore rest on three assumptions, all of which were satisfied in Hartree's numerical solution: (i) there is a singularity at separation; (ii) there is a finite value of u_1 at separation for $y_1 \neq 0$; (iii) $a_2 = \frac{1}{2}$. Related to (iii) Professor Hartree found (empirically) that in his solution $(\partial u_1/\partial y_1)_{y_1=0}$ behaved near $x_1 = 0$ like a multiple of x_1^r, where r is certainly less than 1 and greater than $\frac{1}{4}$. Thus, we must take

$$u_1 = \tfrac{1}{2}y_1^2 + a_3 y_1^3 + \dots \quad \text{at} \quad x_1 = 0, \tag{21}$$

and as a result we find that

$$(\partial u_1/\partial y_1)_{y_1=0} = 2^{\frac{3}{2}}(\alpha_1 x_1^{\frac{1}{2}} + \alpha_2 x_1^{\frac{3}{4}} + \alpha_3 x_1 + \alpha_4 x_1^{\frac{5}{4}} + \dots), \tag{22}$$

where the α's are constants. (The factor $2^{\frac{3}{2}}$ is inserted to conform to the notation in § 2.)

The first purpose of the calculations was to find the connexions between $a_3, a_4, a_5, a_6, \dots$ and $\alpha_1, \alpha_2, \alpha_3, \dots$ and other formulae for u at and near $x_1 = 0$ to see if the results fitted the numerical values for the special solution. There is no mathematical proof that a solution exists with singularities of the type considered near separation, but with the above assumptions it is difficult to see how the solution could be of a different type.

We may remark that in assuming that $a_2 = \frac{1}{2}$, we are in effect assuming that $(\partial^2 u_1/\partial y_1^2)_{y_1=0}$ is continuous at $x_1 = 0$, and then $(\partial^n u_1/\partial y_1^n)_{y_1=0}$ is found to be continuous at $x_1 = 0$ for $n = 1, 2, 3, 4$ and discontinuous for $n = 5$ and 6 and probably for all $n \geqslant 5$, though $\partial^n u_1/\partial y_1^n$ is continuous for $y_1 \neq 0$. More important, it is found that at separation v_1 and $\partial u_1/\partial x_1$ become infinite in such a way that $x_1^{\frac{1}{2}} v_1$ and $x_1^{\frac{1}{2}} \partial u_1/\partial x_1$ have finite non-zero limits as $x_1 \to 0$ for all non-zero y_1. The basic assumptions of boundary-layer theory therefore do not hold at and near separation. Nevertheless, large cross-velocities are to be expected at separation, otherwise the assumptions of boundary-layer theory would not break down.

The formal solution for the motion upstream is found by writing

$$\xi = x_1^{\frac{1}{4}}, \qquad \eta = y_1/2^{\frac{1}{2}} x_1^{\frac{1}{4}}, \tag{23}$$

$$\psi_1 = 2^{\frac{3}{2}}\xi^3[f_0(\eta) + \xi f_1(\eta) + \xi^2 f_2(\eta) + \dots], \tag{24}$$

$$u_1 = 2\xi^2[f_0'(\eta) + \xi f_1'(\eta) + \xi^2 f_2'(\eta) + \dots] \tag{25}$$

in (16), and equating powers of ξ. Since $\psi_1 = 0$ and $u_1 = 0$ at $y_1 = 0$, $f_r(0) = f_r'(0) = 0$, and from the value (21) of u_1 at $x_1 = 0$ we find the condition

$$\lim_{\eta \to \infty} \frac{f_r'}{\eta^{r+2}} = 2^{\frac{1}{2}r} a_{r+2} \quad (r = 0, 1, 2, \dots). \tag{26}$$

The solution for f_r must have a double zero at the origin, and must not involve exponentially large terms as $\eta \to \infty$.

The condition $u_1 \to U_1$ as $y_1 \to \infty$ is satisfied for $x_1 > 0$ if it is satisfied at $x_1 = 0$, i.e. if $u_1 \to 1$ as $y_1 \to \infty$ at $x_1 = 0$.

With $a_2 = \frac{1}{2}$, the solution for f_0 is found to be $f_0 = \eta^3/6$. The solution for f_1 is $f_1 = \alpha_1 \eta^2$, and

$$a_3 = \frac{1}{\sqrt{2}} \lim \frac{f_1'}{\eta^3} = 0. \tag{27}$$

Then
$$f_2 = \alpha_2 \eta^2 - \frac{\alpha_1^2}{15}\eta^5, \tag{28}$$

and
$$a_4 = -\tfrac{1}{6}\alpha_1^2. \tag{29}$$

[The a's and α's are as in (21) and (22).] The equation for f_3 becomes

$$f_3''' - \tfrac{1}{2}\eta^3 f_3'' + \tfrac{7}{2}\eta^2 f_3' - 6\eta f_3 = 5f_1'' f_2 - 7f_1' f_2 + 4f_1 f_2''. \tag{30}$$

The equations for all succeeding f's are non-homogeneous linear equations, with the right-hand side rapidly becoming more and more complicated; thus for f_4 it involves f_1, f_2, f_3, P_1, for f_5 the first four f's, and so on. The complementary functions involve integrals of confluent hypergeometric functions; the particular integrals are very involved. The condition for the absence of exponentially large terms in f_3 is†

$$\alpha_2 = \frac{2^{\frac{1}{4}}\pi^{\frac{1}{2}}}{5(\frac{1}{4}!)^3}\alpha_1^2, \tag{31}$$

and then from (26)
$$a_5 = -\frac{2^{\frac{1}{2}}\pi}{40(\frac{1}{4}!)^2}\alpha_1^3. \tag{32}$$

The condition for the absence of exponentially large terms in f_4 is

$$\alpha_3 = \frac{\pi^3}{400(\frac{1}{4}!)^6}(35 - 8\sqrt{2})\alpha_1^3, \tag{33}$$

and from (26)
$$a_6 = \left(\frac{1}{9} - \frac{7\pi^2}{600(\frac{1}{4}!)^4}\right)\alpha_1^4 - \frac{P_1}{360}. \tag{34}$$

The condition for the absence of exponentially large terms in f_5 gives α_4 as a multiple of α_1^4, though the constant must be found numerically, and then a_7 is found; but the condition for the absence of exponentially large terms in f_6 does *not* give α_5; it requires that

$$\int_0^\infty H_6\left(\eta^2 - \frac{\eta^6}{5} + \frac{\eta^{10}}{180}\right)\exp\left(-\frac{\eta^4}{8}\right)d\eta = 0, \tag{35}$$

where H_6 is a complicated function of η, involving f_5.

Again, α_6 is determined from f_7 in terms of α_1, α_5, and P_1, α_7 from f_8 in terms of α_1, α_5, and P_1, and so on until we come to f_{10}. It is possible, though it has not been proved, that α_5 is determined from f_{10}, α_9 from f_{14},

† $x!$ is written for $\Gamma(x+1)$.

and so on. If so, then only α_1 remains to be determined, and a_4 (and therefore α_1) is probably determined by the condition $u_1 \to 1$ as $y_1 \to \infty$ at $x_1 = 0$. If so the whole solution is determined at separation. In fact, if it is true that all the other constants are determinate in terms of a_4 and the P's, there is a solution only if it is possible to choose a_4 so that the condition $u_1 \to 1$ as $y_1 \to \infty$ at $x_1 = 0$ is satisfied. Unless this condition is satisfied for every value of a_4, it will presumably fix a_4 in terms of the P's. If a_4 is so fixed, the non-dimensional velocity distribution at separation, $x_1 = 0$, and just upstream of separation, for small positive values of x_1, is fixed in terms of the P's. Suppose now we have a problem in which U is a given decreasing function of x, and u a given function of y for some $x < x_s$. There are some U's for which the P's are independent of x_s; otherwise they vary with x_s. When separation takes place, the non-dimensional velocity distribution at and near separation is independent of the initial distribution of u for the former values of U, and for the others it is the same for all initial distributions of u for which separation takes place at the same value of x. This suggests that what has been found is an *asymptotic* solution at and near separation, and that the full non-dimensional solutions in the above cases all behave asymptotically in the same way near separation.

It appears that the singularity at separation is due to the non-linear character of the equations. It is possible to simulate the phenomenon of separation by a linear system of equations, and there is then no singularity at separation. For example, the solution of

$$\frac{\partial u}{\partial t} = -1 + \frac{\partial^2 u}{\partial y^2}, \quad u = 1 \text{ at } t = 0, \quad u = 0 \text{ at } y = 0, \quad u \text{ finite as } y \to \infty, \tag{36}$$

is

$$u = \tfrac{1}{2}y^2 + (1 - t - \tfrac{1}{2}y^2)\operatorname{erf}\frac{y}{2\sqrt{t}} - y\sqrt{\frac{t}{\pi}}\exp\left(-\frac{y^2}{4t}\right), \tag{37}$$

where

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int\limits_0^x e^{-w^2}\,dw, \tag{38}$$

so

$$(\partial u/\partial y)_{y=0} = 0 \quad \text{at} \quad t = \tfrac{1}{2}. \tag{39}$$

We should remark also that in equations that are linear but are otherwise similar to the equations here considered (the equation for the temperature in the theory of the conduction of heat,† for example), if we attempt to work backwards (i.e. to solve for negative time) from a

† The relationship of the boundary-layer equations to the equation of heat conduction has been stressed by Prandtl (**2**) (loc. cit., pp. 79, 80) in connexion with difficulties to be expected when $u \leqslant 0$.

singularity we encounter exponentially large terms. With given initial and boundary conditions, however, the solution for such a linear equation is free from singularities for positive non-zero time, whereas the basis of the present discussion is the assumption that singularities may occur at separation in the solution of the non-linear equations considered.

The special case considered by Professor Hartree is one in which by a correct choice of scale the P's may be made independent of x_s. There is always the possibility, therefore, that the occurrence of a singularity at separation is restricted to such cases. Another possibility is that a singularity will always occur except for certain special pressure variations in the neighbourhood of separation, and that, experimentally, whatever we may do, the pressure variations near separation will always be such that no singularity will occur.

It is a necessary consequence of the discussion of the motion upstream of separation that a_4 is negative or zero. Professor Hartree finds a negative a_4 from his special numerical solution. When we consider the motion *downstream* of separation in a similar way, we find that when a_4 is negative the solution downstream is not *real*. When there is a singularity at separation there is no real solution at all farther downstream. When $a_4 = 0$ there is a solution downstream, but then we have a case in which the whole solution is free from singularities. These cases include that in which $(\partial u/\partial y)_{y=0} = 0$ for all x. There must, of course, be restrictions on the pressure distributions in order that this should happen, and these conditions arise from the condition that $u_1 \to 1$ as $y_1 \to \infty$ at $x_1 = 0$. This our method does not permit us to discuss, but one solution (due to Falkner and Skan, **4**†) is known in which $U = cx^m$, $m = -0{\cdot}0904$ approximately.

As far as numerical values are concerned, and comparison with the computed values, Professor Hartree fitted his solution to the formulae here obtained, and considered that he had obtained a reasonable fit. The matter has been recently reconsidered by Mr. C. W. Jones, who has tabulated f_3, f_4, f_5 and has found that within the accuracy of his computation, the integral condition (35) for the absence of exponentially large terms in f_6 is satisfied.

Mr. Jones has compared the skin-friction, the velocity distributions at separation not far from the wall, the transition to main-stream conditions at separation, and the velocity distribution just downstream of separation (at $(8\beta_1/\beta_0)x = 0{\cdot}956$, where $U = \beta_0 - \beta_1 x$ and $(8\beta_1/\beta_0)x_s = 0{\cdot}959$). A satisfactory fit is obtained with α_1 about 0·47 or 0·48. A satisfactory transition to main-stream conditions seems to be obtained, but it is not sensitive to changes in α_1.

† See also (**5**).

If we assume that Mr. Jones's numerical work is sufficient to answer certain questions, and to make it plausible that our formulae fit the solution in the case considered, we still do not know for certain that α_5, α_9,... are determined from the equations for f_9, f_{13},..., and, if they are, that a_4 (and therefore α_1) can be determined from the condition $u_1 \to 1$ as $y_1 \to \infty$ at $x_1 = 0$. It is clear that an adequate discussion is not possible by the method used here. Three more important questions also remain:

(1) Is it correct that the formulae represent an asymptotic solution at and near separation?

(2) What are the most general restrictions on the pressure distribution in order that solutions should exist for which $(\partial u/\partial y)_{y=0} = 0$ for all x?

(3) A singularity when $U = \beta_0 - \beta_1 x$ being assumed, is the occurrence of a singularity restricted to cases in which the P's are independent of x_s? Or does a singularity always occur unless $(\partial u/\partial y)_{y=0} = 0$ for all x? Or does a singularity always occur except for certain special pressure distributions near separation, and are experimental pressure distributions always of the special type?

It should be remarked that although there is a certain physical plausibility in the notion that large cross-velocities should occur at separation, the existing experimental information is insufficient to settle the question.

The work described may be summed up by saying that it throws doubt on the validity of the boundary-layer equations at and near separation on the upstream side, and also downstream of separation; inferences from these equations in these regions, which are fairly common in the literature, are therefore also in doubt; mathematically the work suggests that singularities may arise in the solution of non-linear parabolic equations, due to their non-linearity; and formulae have been found which may help computers of laminar boundary layers, who desire more than a rough solution, to have an end-point at which to aim.

2. The solution upstream

Substitute (23), (24), and (17) into (16), and equate coefficients of powers of ξ. The equation for f_0, obtained from the coefficient of ξ^0, is

$$f_0''' - 3f_0 f_0'' + 2f_0'^2 = 1. \tag{40}$$

Since $\psi_1 = 0$ and $u_1 = 0$ at $y_1 = 0$,

$$f_r(0) = f_r'(0) = 0 \quad (r = 0, 1, 2,...). \tag{41}$$

When $\xi \to 0$, $\eta \to \infty$ if $y_1 \neq 0$, and since $2^{\frac{1}{2}}\xi\eta = y_1$, $\lim_{\xi \to 0} u_1$ is given by (21) if

$$\lim_{\eta \to \infty} \frac{f_r'}{\eta^{r+2}} = 2^{\frac{1}{2}r} a_{r+2} \quad (r = 0, 1, 2,...). \tag{42}$$

The condition that the velocity should pass over smoothly into the velocity of the main stream will be considered in § 3, when the solution for large values of $y_1/x_1^{\frac{1}{4}}$ is considered.

Since $a_2 = \frac{1}{2}$, the solution for f_0 is

$$f_0 = \eta^3/6. \tag{43}$$

The equation for f_r is then found to be

$$f_r''' - \tfrac{1}{2}\eta^3 f_r'' + \tfrac{1}{2}(r+4)\eta^2 f_r' - (r+3)\eta f_r = G_r, \tag{44}$$

where

$$G_1 = 0, \qquad G_2 = 4f_1 f_1'' - 3f_1'^2, \tag{45}$$

and for $r \geqslant 2$,

$$G_r = \sum_{s=1}^{r-1} [(r-s+3)f_s'' f_{r-s} - (r-s+2)f_s' f_{r-s}'] + P_{r/4}, \tag{46}$$

$P_{r/4}$ being put equal to zero except when $\frac{1}{4}r$ is integral. The solution for f_1 with a double zero at the origin is

$$f_1 = \alpha_1 \eta^2, \tag{47}$$

where α_1 is a constant; hence from (42),

$$a_3 = \frac{1}{\sqrt{2}} \lim_{\eta\to\infty} \frac{f_1'}{\eta^3} = 0. \tag{48}$$

The solution for f_2 with a double zero at the origin is now found to be

$$f_2 = \alpha_2 \eta^2 - \frac{\alpha_1^2}{15}\eta^5, \tag{49}$$

where α_2 is a constant; hence from (42)

$$a_4 = -\tfrac{1}{6}\alpha_1^2. \tag{50}$$

In order to write down the general solution for f_3 with a double zero at the origin, and to consider its behaviour as $\eta \to \infty$, some discussion is necessary of the complementary functions, and it is advisable to break off and discuss generally the complementary functions of the equation for f_r.

Three independent complementary functions are η^2, g_r, and h_r, where, if the function ${}_1F_1(a, b, x)$ is defined by

$${}_1F_1(a,b,x) = 1 + \frac{a}{1\,.\,b}x + \frac{a(a+1)}{2!\,b(b+1)}x^2 + \frac{a(a+1)(a+2)}{3!\,b(b+1)(b+2)}x^3 + \ldots, \tag{51}$$

then†

$$g_r = -\sum_{m=0}^{\infty} \frac{(m-\frac{3}{2}-\frac{1}{4}r)!\,\frac{1}{4}!\,\eta^{4m+1}}{m!\,(-\frac{3}{2}-\frac{1}{4}r)!\,(m+\frac{1}{4})!\,8^m(4m-1)}$$

$$= \eta - \eta^2 \int_0^\eta \eta^{-2}\{{}_1F_1(-\tfrac{1}{2}-\tfrac{1}{4}r, \tfrac{5}{4}, \eta^4/8) - 1\}\,d\eta, \tag{52}$$

† $x!$ is written for $\Gamma(x+1)$, as before.

and
$$h_r = -\sum_{m=0}^{\infty} \frac{(m-\frac{7}{4}-\frac{1}{4}r)!\,(-\frac{1}{4})!\,\eta^{4m}}{m!\,(-\frac{7}{4}-\frac{1}{4}r)!\,(m-\frac{1}{4})!\,8^m(2m-1)}$$
$$= 1-2\eta^2\int\limits_0^{\eta} \eta^{-3}\{{}_1F_1(-\tfrac{3}{4}-\tfrac{1}{4}r,\tfrac{3}{4},\eta^4/8)-1\}\,d\eta. \tag{53}$$

The series for g_r terminates when $r = 4m+2$, and that for h_r terminates when $r = 4m+1$, m being a positive integer or zero.

As regards asymptotic expansions, in addition to the solution η^2 the equation (44) with G_r put equal to zero has two solutions whose asymptotic expansions for large η commence with multiples of
$$\eta^{r+3} \quad \text{and of} \quad \eta^{-(r+10)}\exp(\eta^4/8)$$
respectively.

When x is large and positive (**6**),†
$${}_1F_1(a,b,x) \sim \frac{(b-1)!}{(a-1)!}e^x x^{a-b}\times$$
$$\times\left\{1+\frac{(b-a)(1-a)}{x}+\frac{(b-a)(b-a+1)(1-a)(2-a)}{2!\,x^2}+\ldots\right\} \tag{54}$$
so
$${}_1F_1(-\tfrac{1}{2}-\tfrac{1}{4}r,\tfrac{5}{4},\eta^4/8) \sim \frac{2^{(3r+13)/4}(-\frac{3}{4})!}{(-\frac{3}{2}-\frac{1}{4}r)!}\exp(\eta^4/8)\eta^{-(r+7)}\{1+\ldots\} \quad (r \neq 4m+2) \tag{55}$$
and
$${}_1F_1(-\tfrac{3}{4}-\tfrac{1}{4}r,\tfrac{3}{4},\eta^4/8) \sim -\frac{2^{(3r+10)/4}(-\frac{5}{4})!}{(-\frac{7}{4}-\frac{1}{4}r)!}\exp(\eta^4/8)\eta^{-(r+6)}\{1+\ldots\} \quad (r \neq 4m+1). \tag{56}$$
Hence
$$g_r \sim -\frac{2^{(3r+17)/4}(-\frac{3}{4})!}{(-\frac{3}{2}-\frac{1}{4}r)!}\exp(\eta^4/8)\eta^{-(r+10)}\{1+\ldots\} \quad (r \neq 4m+2) \tag{57}$$
and
$$h_r \sim \frac{2^{(3r+18)/4}(-\frac{5}{4})!}{(-\frac{7}{4}-\frac{1}{4}r)!}\exp(\eta^4/8)\eta^{-(r+10)}\{1+\ldots\} \quad (r \neq 4m+1). \tag{58}$$

Exponentially large terms must not occur in the solution for f_r, so when $r \neq 4m+1$ or $4m+2$, g_r and h_r must occur in the combination
$$(-\tfrac{5}{4})!\,(-\tfrac{3}{2}-\tfrac{1}{4}r)!\,g_r+2^{-\frac{1}{4}}(-\tfrac{3}{4})!\,(-\tfrac{7}{4}-\tfrac{1}{4}r)!\,h_r. \tag{59}$$
But‡
$$(a-1)!\,(-b)!\,{}_1F_1(a,b,x)+(a-b)!\,(b-2)!\,x^{1-b}{}_1F_1(a+1-b,2-b,x)$$
$$\sim (a-1)!\,(a-b)!\,x^{-a}\left\{1-\frac{a(a+1-b)}{x}+\frac{a(a+1)(a+1-b)(a+2-b)}{2!\,x^2}-\ldots\right\}; \tag{60}$$

† The formula is on p. 258 of ref. (**6**).
‡ Barnes (**5**), op. cit., p. 259.

hence

$$
\begin{aligned}
&(-\tfrac{5}{4})!\,(-\tfrac{3}{2}-\tfrac{1}{4}r)!\,{}_1F_1(-\tfrac{1}{2}-\tfrac{1}{4}r,\tfrac{5}{4},\eta^4/8)+\\
&\qquad+8^{\frac14}(-\tfrac{3}{4})!(-\tfrac{7}{4}-\tfrac{1}{4}r)!\,\eta^{-1}{}_1F_1(-\tfrac{3}{4}-\tfrac{1}{4}r,\tfrac{3}{4},\eta^4/8)\\
&\quad\sim 8^{-(r+2)/4}(-\tfrac{3}{2}-\tfrac{1}{4}r)!\,(-\tfrac{7}{4}-\tfrac{1}{4}r)!\,\eta^{r+2}\Big\{1-\frac{(2+r)(3+r)}{2\eta^4}+\\
&\qquad+\frac{(2+r)(2-r)(3+r)(1-r)}{2\,.\,4\,.\,\eta^8}-\frac{(2+r)(2-r)(6-r)(3+r)(1-r)(5-r)}{2\,.\,4\,.\,6\,.\,\eta^{12}}+\dots\Big\}
\end{aligned}
$$

$$(r \neq 4m+1 \text{ or } 4m+2) \tag{61}$$

and

$$
\begin{aligned}
&(-\tfrac{5}{4})!\,(-\tfrac{3}{2}-\tfrac{1}{4}r)!\,g_r+2^{-\frac14}(-\tfrac{3}{4})!\,(-\tfrac{7}{4}-\tfrac{1}{4}r)!\,h_r\\
&\quad=(-\tfrac{5}{4})!(-\tfrac{3}{2}-\tfrac{1}{4}r)!\,\eta+2^{-\frac14}(-\tfrac{3}{4})!\,(-\tfrac{7}{4}-\tfrac{1}{4}r)!-\\
&\qquad-\eta^2\int\limits_0^\eta \eta^{-2}\{(-\tfrac{5}{4})!\,(-\tfrac{3}{2}-\tfrac{1}{4}r)!\,[{}_1F_1(-\tfrac{1}{2}-\tfrac{1}{4}r,\tfrac{5}{4},\eta^4/8)-1]+\\
&\qquad\qquad+8^{\frac14}(-\tfrac{3}{4})!\,(-\tfrac{7}{4}-\tfrac{1}{4}r)!\,\eta^{-1}[{}_1F_1(-\tfrac{3}{4}-\tfrac{1}{4}r,\tfrac{3}{4},\eta^4/8)-1]\}\,d\eta\\
&\quad\sim -8^{-(r+2)/4}(-\tfrac{3}{2}-\tfrac{1}{4}r)!\,(-\tfrac{7}{4}-\tfrac{1}{4}r)!\Big\{\frac{\eta^{r+3}}{r+1}-\frac{(2+r)(3+r)\eta^{r-1}}{2(r-3)}+\\
&\qquad+\frac{(2+r)(2-r)(3+r)(1-r)\eta^{r-5}}{2\,.\,4\,.\,(r-7)}-\\
&\qquad-\frac{(2+r)(2-r)(6-r)(3+r)(1-r)(5-r)\eta^{r-9}}{2\,.\,4\,.\,6\,.\,(r-11)}+\dots\Big\}+\\
&\qquad+\text{constant}\,.\,\eta^2 \qquad\qquad (r=4m).
\end{aligned}\tag{62}
$$

When $r = 4m+3$ the term with a zero denominator must be replaced by

$$
\begin{aligned}
&(-1)^m 8^{-m-5/4}(-m-\tfrac{9}{4})!\,(-m-\tfrac{5}{2})!\times\\
&\qquad\times\frac{5\,.\,9\,.\,13\dots(4m+5)\,.\,3\,.\,5\,.\,7\dots(2m+3)}{(m+1)!}\eta^2\log\eta,
\end{aligned}
$$

which reduces to

$$\frac{(-1)^{m+1}\,.\,2^{\frac14}\,.\,\pi^{\frac12}\,.\,(-\tfrac{5}{4})!}{(m+1)!}\eta^2\log\eta. \tag{63}$$

(62) may be verified and the constant multiplier of η^2 found by considering

$$\frac{1}{2\pi i}\int(-s-1)!\,(-\tfrac{5}{4}-s)!\,(s-\tfrac{3}{2}-\tfrac{1}{4}r)!\,\frac{\eta^{4s+1}}{8^s(4s-1)} \tag{64}$$

taken round a contour consisting of the straight line from $-N-\frac{1}{2}-\infty i$ to $-N-\frac{1}{2}+\infty i$ and the part of a circle of infinite radius to the right of the line. When $r \neq 4m+3$ the constant multiple of η^2 in (62) is found to be

$$2^{-7/4}\pi^{\frac12}(-\tfrac{5}{4})!\,(-\tfrac{5}{4}-\tfrac{1}{4}r)!\,\eta^2. \tag{65}$$

When $r = 4m+3$ the term with a zero denominator in (62) must be omitted, and (65) replaced by

$$\frac{(-1)^{m+1}\pi^{\frac{1}{2}}(-\frac{5}{4})!}{2^{7/4}(m+1)!}\eta^2\left[\log\frac{\eta^4}{8}-\mathfrak{F}(-\tfrac{5}{4})-\mathfrak{F}(-\tfrac{3}{2})+\mathfrak{F}(m+1)\right], \tag{66}$$

where
$$\mathfrak{F}(z) = \frac{d}{dz}\log z!. \tag{67}$$

The multiplier of $\eta^2\log\eta$ agrees with that in (63). If γ is Euler's constant (equal to $0{\cdot}5772\ldots$)

$$\left.\begin{aligned}\mathfrak{F}(m+1) &= 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{m+1}-\gamma,\\ \mathfrak{F}(-\tfrac{5}{4}) &= \tfrac{1}{2}\pi+4-3\log 2-\gamma,\\ \mathfrak{F}(-\tfrac{3}{2}) &= 2-2\log 2-\gamma,\end{aligned}\right\} \tag{68}$$

so (66) is equal to

$$\frac{(-1)^{m+1}\pi^{\frac{1}{2}}(-\frac{5}{4})!}{2^{7/4}(m+1)!}\eta^2\left[4\log\eta+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{m+1}+2\log 2+\gamma-\tfrac{1}{2}\pi-5\right]. \tag{69}$$

In particular, since

$$(-\tfrac{5}{4})! = -\frac{2^{\frac{1}{2}}.\pi}{\frac{1}{4}!},\quad (-\tfrac{5}{2})! = \frac{4\pi^{\frac{1}{2}}}{3},\quad (-\tfrac{9}{4})! = \frac{4.2^{\frac{1}{2}}.\pi}{5.\frac{1}{4}!},\quad (-\tfrac{11}{4})! = \frac{64.\frac{1}{4}!}{21}, \tag{70}$$

it follows from (62), (65), and (69) that

$$\begin{aligned}h_3-\frac{3.2^{\frac{1}{4}}.\pi^{\frac{3}{2}}}{10.(\frac{1}{4}!)^3}g_3 \sim -\frac{\pi}{160(\frac{1}{4}!)^2}&\left\{\eta^6-\frac{15}{2!}\frac{1}{\eta^2}+\frac{15.1.3}{2.3!}\frac{1}{\eta^6}-\right.\\ &\left.-\frac{15.1.3.3.7}{3.4!}\frac{1}{\eta^{10}}+\ldots\right\}+\frac{3\pi}{32(\frac{1}{4}!)^2}\eta^2[4\log\eta+2\log 2+\gamma-\tfrac{1}{2}\pi-5]\end{aligned} \tag{71}$$

and

$$\begin{aligned}h_4-\frac{7.2^{\frac{3}{4}}.\pi^{\frac{3}{2}}}{64(\frac{1}{4}!)^3}g_4 \sim -\frac{\pi^{\frac{1}{2}}}{48.2^{\frac{1}{4}}.\frac{1}{4}!}&\left\{\frac{\eta^7}{5}-\frac{3.7}{1!}\eta^3-\frac{3.1.7.3}{3.2!}\frac{1}{\eta}+\right.\\ &+\frac{3.1.1.7.3.1}{7.3!}\frac{1}{\eta^5}-\frac{3.1.1.3.7.3.1.5}{11.4!}\frac{1}{\eta^9}+\\ &\left.+\frac{3.1.1.3.5.7.3.1.5.9}{15.5!}\frac{1}{\eta^{13}}-\ldots\right\}-\frac{21.2^{\frac{1}{4}}.\pi^{\frac{3}{2}}}{640(\frac{1}{4}!)^4}\eta^2.\end{aligned} \tag{72}$$

We now return to the equation for f_3, which is

$$f_3'''-\tfrac{1}{2}\eta^3 f_3''+\tfrac{7}{2}\eta^2 f_3'-6\eta f_3 = -10\alpha_1\alpha_2\eta^2-\tfrac{4}{3}\alpha_1^3\eta^5. \tag{73}$$

The general integral with a double zero at the origin is

$$f_3 = \alpha_3\eta^2+4\alpha_1\alpha_2(\eta-g_3)-\tfrac{8}{3}\alpha_1^3(1+\tfrac{1}{4}\eta^4-h_3), \tag{74}$$

where α_3 is a constant. In order that exponentially large terms should

not occur in f_3, g_3 and h_3 must appear in (74) in the same combination as in (71). Hence we must have

$$\alpha_2 = \frac{2^{\frac14}.\pi^{\frac32}}{5(\frac14 !)^3}\alpha_1^2 \tag{75}$$

and

$$f_3 = \alpha_3\eta^2 - \tfrac83\alpha_1^3\left\{1+\tfrac14\eta^4 - h_3 - \frac{3.2^{\frac14}.\pi^{\frac32}}{10(\frac14 !)^3}(\eta-g_3)\right\}$$

$$= \alpha_3\eta^2 - \tfrac83\alpha_1^3\left\{\sum_{m=2}^{\infty}\frac{(m-\frac52)!\,(-\frac14 !)\,\eta^{4m}}{m!\,(-\frac52)!\,(m-\frac14)!\,8^m(2m-1)} - \right.$$

$$\left. -\frac{3.2^{\frac14}.\pi^{\frac32}}{10(\frac14 !)^3}\sum_{m=1}^{\infty}\frac{(m-\frac94)!\,\frac14 !\,\eta^{4m+1}}{m!\,(-\frac94)!\,(m+\frac14)!\,8^m(4m-1)}\right\}. \tag{76}$$

For large values of η,

$$f_3 \sim -\frac{\pi}{60(\frac14 !)^2}\alpha_1^3\left\{\eta^6 - \frac{15}{2!}\frac{1}{\eta^2} + \frac{15.1.3}{2.3!}\frac{1}{\eta^6} - \frac{15.1.3.3.7}{3.4!}\frac{1}{\eta^{10}} + \dots\right\} +$$

$$+ \frac{\pi}{4(\frac14 !)^2}\alpha_1^3\eta^2[4\log\eta + 2\log 2 + \gamma - \tfrac12\pi - 5] +$$

$$+ \alpha_3\eta^2 - \tfrac83\alpha_1^3\left[1+\tfrac14\eta^4 - \frac{3.2^{\frac14}.\pi^{\frac32}}{10(\frac14 !)^3}\eta\right]. \tag{77}$$

Hence, from (42),
$$a_5 = -\frac{2^{\frac12}.\pi}{40(\frac14 !)^2}\alpha_1^3. \tag{78}$$

We now turn to the equation for f_4, which on substituting in (44) we find to be

$$f_4''' - \tfrac12\eta^3 f_4'' + 4\eta^2 f_4' - 7\eta f_4 = P_1 - 6(\alpha_2^2 + 2\alpha_1\alpha_3)\eta^2 - 2\alpha_1^2\alpha_2\eta^5 -$$

$$-\tfrac{32}{3}\alpha_1^4\left\{\sum_{m=2}^{\infty}\frac{(16m^2-20m+3)(m-\frac52)!\,(-\frac14)!\,\eta^{4m}}{m!\,(-\frac52)!\,(m-\frac14)!\,8^m(2m-1)} - \right.$$

$$\left. -\frac{3.2^{\frac14}.\pi^{\frac32}}{10(\frac14 !)^3}\sum_{m=1}^{\infty}\frac{(16m^2-12m-1)(m-\frac94)!\,\frac14 !\,\eta^{4m+1}}{m!\,(-\frac94)!\,(m+\frac14)!\,8^m(4m-1)}\right\}. \tag{79}$$

The general solution of (79) with a double zero at the origin is (with α_4 denoting a constant)

$$f_4 = \alpha_4\eta^2 + \frac{P_1}{6}\left(\eta^3 - \frac{\eta^7}{105}\right) + 2(\alpha_2^2 + 2\alpha_1\alpha_3)(\eta - g_4) - \tfrac{16}{7}\alpha_1^2\alpha_2\left(1 + \frac{7\eta^4}{24} - h_4\right) -$$

$$-\tfrac{32}{3}\alpha_1^4\left\{L + \frac{3.2^{\frac14}.\pi^{\frac32}}{28.(\frac14 !)^3}\left(1 + \frac{7\eta^4}{24} - h_4\right)\right\}, \tag{80}$$

where

$$L = \sum_{m=2}^{\infty} \frac{(m-\frac{5}{2})!\,(-\frac{1}{4})!\,(2m-3)\eta^{4m+3}}{m!\,(-\frac{5}{2})!(m-\frac{1}{4})!\,8^m(4m+2)(4m+3)} - \\ -\frac{3\,.\,2^{\frac{1}{4}}\,.\,\pi^{\frac{3}{2}}}{40(\frac{1}{4}!)^3} \sum_{m=1}^{\infty} \frac{(m-\frac{9}{4})!\,\frac{1}{4}!\,(4m-5)\eta^{4m+4}}{m!\,(-\frac{9}{4})!\,(m+\frac{1}{4})!\,8^m(4m+3)(4m+4)}. \tag{81}$$

Now

$$\frac{d^2L}{d\eta^2} = \eta^8 \frac{d}{d\eta}\left\{\frac{1}{2\eta^6}[{}_1F_1(-\tfrac{3}{2}, \tfrac{3}{4}, \eta^4/8)-1+\tfrac{1}{4}\eta^4] - \right. \\ \left. -\frac{3\,.\,2^{\frac{1}{4}}\,.\,\pi^{\frac{3}{2}}}{40(\frac{1}{4}!)^3}\,\frac{1}{\eta^5}[{}_1F_1(-\tfrac{5}{4}, \tfrac{5}{4}, \eta^4/8)-1]\right\}, \tag{82}$$

and since, from (61) and (70)

$$\frac{1}{2\eta^6}\,{}_1F_1(-\tfrac{3}{2}, \tfrac{3}{4}, \eta^4/8) - \frac{3\,.\,2^{\frac{1}{4}}\,.\,\pi^{\frac{3}{2}}}{40(\frac{1}{4}!)^3}\,\frac{1}{\eta^5}\,{}_1F_1(-\tfrac{5}{4}, \tfrac{5}{4}, \eta^4/8) \\ \sim \frac{3\pi}{32(\frac{1}{4}!)^2}\left\{\frac{1}{15} - \frac{1}{\eta^4} + \frac{1}{2!\,\eta^8} - \frac{3}{3!\,\eta^{12}} + \frac{3\,.\,7\,.\,1\,.\,3}{4!\,\eta^{16}} - \frac{3\,.\,7\,.\,11\,.\,1\,.\,3\,.\,5}{5!\,\eta^{20}} + \ldots\right\}, \tag{83}$$

the asymptotic expansion of L may be deduced apart from an additive constant and an additive multiple of η. The resulting expression may be checked, and the constant and the multiple of η determined, by considering

$$\frac{1}{2\pi i}\int (-s-1)!\,(-s-\tfrac{5}{4})!\,(s-\tfrac{9}{4})!\,\frac{(4s-5)\eta^{4s+4}}{8^s(4s+3)(4s+4)}\,ds \tag{84}$$

taken round a contour consisting of the straight line from $-N-\frac{1}{2}-\infty i$ to $-N-\frac{1}{2}+\infty i$ and the part of a circle of infinite radius to the right of the line. After some calculation it is found that†

$$L \sim -\frac{\eta^7}{168} - \frac{2^{\frac{1}{4}}\,.\,\pi^{\frac{3}{2}}}{32(\frac{1}{4}!)^3}\eta^4 + \frac{\eta^3}{2} - \frac{3\pi}{32(\frac{1}{4}!)^2}\eta\{4\log\eta+2\log 2+\gamma-\tfrac{1}{2}\pi-3\} - \\ -\frac{3\,.\,2^{\frac{1}{4}}\,.\,\pi^{\frac{3}{2}}}{20(\frac{1}{4}!)^3} + \frac{3\pi}{32(\frac{1}{4}!)^2}\left\{\frac{\eta^5}{5} + \frac{1}{1\,.\,2!}\,\frac{1}{\eta^3} - \frac{1\,.\,3\,.\,3}{2\,.\,3!}\,\frac{1}{\eta^7} + \frac{1\,.\,3\,.\,5\,.\,3\,.\,7}{3\,.\,4!}\,\frac{1}{\eta^{11}} - \ldots\right\}. \tag{85}$$

As before γ denotes Euler's constant, and use has been made of (70), of the first equation of (68) with $m = 1$, and of the formulae

$$\mathfrak{F}(-\tfrac{1}{4}) = \tfrac{1}{2}\pi - 3\log 2 - \gamma, \qquad \mathfrak{F}(-\tfrac{1}{2}) = -2\log 2 - \gamma. \tag{86}$$

The asymptotic expansions of all the terms in the expression for f_4 in (80) are now known. In order that exponentially large terms should be absent from the asymptotic expansion of f_4, g_4 and h_4 must occur in (80) in the same combination as in (72). The terms containing g_4 and h_4 in (80) are

$$-2(\alpha_2^2+2\alpha_1\,\alpha_3)g_4 + \left(\tfrac{16}{7}\alpha_1^2\,\alpha_2 + \frac{8\,.\,2^{\frac{1}{4}}\,.\,\pi^{\frac{3}{2}}}{7(\frac{1}{4}!)^3}\alpha_1^4\right)h_4. \tag{87}$$

† I am indebted to Mr. C. W. Jones for the correction of three errors in (85) as first written.

Compare with (72), and use the value of α_2 from (75). We must have

$$2(\alpha_2^2+2\alpha_1\alpha_3) = \frac{7\pi^3}{20(\frac{1}{4}!)^6}\alpha_1^4 \tag{88}$$

and

$$\alpha_3 = \frac{\pi^3}{400(\frac{1}{4}!)^6}(35-8.2^{\frac{1}{2}})\alpha_1^3. \tag{89}$$

Substituting for the coefficients of g_4 and h_4 in (80) from (88) and (75), we find that

$$f_4 = \alpha_4\eta^2+\frac{P_1}{6}\left(\eta^3-\frac{\eta^7}{105}\right)-\tfrac{32}{3}\alpha_1^4 L+$$

$$+\frac{8.2^{\frac{1}{4}}.\pi^{\frac{3}{2}}}{5(\frac{1}{4}!)^3}\alpha_1^4\left\{\left(h_4-1-\frac{7\eta^4}{24}\right)-\frac{7.2^{\frac{3}{4}}.\pi^{\frac{3}{2}}}{64.(\frac{1}{4}!)^3}(g_4-\eta)\right\}. \tag{90}$$

The series expansions of L, g_4, and h_4 are given by (81), (52), and (53), so that the series expansion of f_4 is easily written down. Its asymptotic expansion is found from (72) and (85) to be

$$f_4 \sim \left(\frac{4\alpha_1^4}{63}-\frac{P_1}{630}\right)\eta^7-\frac{2^{\frac{5}{4}}.\pi^{\frac{3}{2}}}{15(\frac{1}{4}!)^3}\alpha_1^4\eta^4+\left(\frac{P_1}{6}-\frac{16\alpha_1^4}{3}\right)\eta^3+$$

$$+\left(\alpha_4-\frac{21.2^{\frac{3}{2}}.\pi^4}{400(\frac{1}{4}!)^7}\alpha_1^4\right)\eta^2+\frac{4\pi}{(\frac{1}{4}!)^2}\alpha_1^4\eta\log\eta+$$

$$+\frac{\pi}{(\frac{1}{4}!)^2}\alpha_1^4\eta\left[2\log 2+\gamma-\tfrac{1}{2}\pi-3+\frac{7\pi^2}{20(\frac{1}{4}!)^4}\right]-$$

$$-\frac{\pi^2}{30(\frac{1}{4}!)^4}\alpha_1^4\left\{\frac{\eta^7}{5}-\frac{3.7}{1!}\eta^3-\frac{3.1.7.3}{1!}\frac{1}{\eta}+\frac{3.1.1.7.3.1}{7.3!}\frac{1}{\eta^5}-\right.$$

$$\left.-\frac{3.1.1.3.7.3.1.5}{11.4!}\frac{1}{\eta^9}+\dots\right\}-$$

$$-\frac{\pi}{(\frac{1}{4}!)^2}\alpha_1^4\left\{\frac{\eta^5}{5}+\frac{1}{1.2!}\frac{1}{\eta^3}-\frac{1.3.3}{2.3!}\frac{1}{\eta^7}+\frac{1.3.5.3.7}{3.4!}\frac{1}{\eta^{11}}-\dots\right\}. \tag{91}$$

It follows from (42) that

$$a_6 = \alpha_1^4\left[\frac{1}{9}-\frac{7\pi^2}{600(\frac{1}{4}!)^4}\right]-\frac{P_1}{360}. \tag{92}$$

The analytical discussion of f_4 is now complete. The equations for f_5, f_6, etc., have not been completely solved, but some discussion of these equations will now be given.

In addition to the complementary function $\alpha_r\eta^2$, the equation for f_r has one complementary function whose asymptotic expansion for large positive values of η commences with a multiple of $\eta^{-(r+10)}\exp(\eta^4/8)$, and another complementary function whose asymptotic expansion commences with a

multiple of η^{r+3} followed by a multiple of η^{r-1}. For the solution to be successful, exponentially large terms must not occur in the asymptotic expansion of f_r, which must begin with a multiple of η^{r+3}. For $r \leqslant 4$ this condition is satisfied, and, moreover, the next term in the asymptotic expansion is a multiple of η^{r+1}. Assume for the moment that these statements are correct for $r \leqslant n-1$. Consider the asymptotic expansion of G_n, the left-hand side of the equation for f_n in (44). The terms of highest degree occurring could be multiples of η^{n+4}, but it is not difficult to prove that the terms in η^{n+4} always cancel, and that in fact the asymptotic expansion of G_n begins with a multiple of η^{n+2}. It follows that the equation for f_n has a particular integral, which we denote by I, whose asymptotic expansion begins with a multiple of η^{n+1}. Any particular integral may be expressed as the sum of I and multiples of the complementary functions. The particular integral with a double zero at the origin is indeterminate only to the extent of an additive multiple of η^2, and what is required is that it should not involve the complementary function that is exponentially large at infinity. If the presence of the exponentially large complementary function can be avoided, the asymptotic expansion of the solution with a double zero at the origin will begin with a multiple of η^{n+3} (from the other complementary function), followed by a multiple of η^{n+1} (from I). Hence by induction it will be true generally that the asymptotic expansion of f_r will begin with a multiple of η^{r+3}, followed by a multiple of η^{r+1}.

Now f_{n-1} contains a term $\alpha_{n-1}\eta^2$, where α_{n-1} is a constant which is undetermined at that stage of the solution at which we solve for f_{n-1}. The only terms in G_n containing α_{n-1} arise from the terms in the expression for G_n which contain f_{n-1} or its derivatives, and it is easy to see that the sum of the terms in G_n containing α_{n-1} is $-(2n+4)\alpha_1\alpha_{n-1}\eta^2$. The corresponding term in the solution for f_n with a double zero at the origin is $4\alpha_1\alpha_{n-1}(\eta-g_n)$. Unless $n = 4m+2$, where m is a positive integer or zero, the asymptotic expansion of g_n involves exponentially large terms. Other exponentially large terms occurring in the solution for f_n can arise only from multiples of g_n or (when $n \neq 4m+1$) from multiples of h_n. Since a suitable combination of h_n and g_n has an asymptotic expansion devoid of exponentially large terms, and this combination involves h_n unless $n = 4m+2$, it follows that when $n \neq 4m+2$ the presence of exponentially large terms in the solution for f_n can always be avoided by a suitable choice of α_{n-1}. In this way, when $n \neq 4m+2$, α_{n-1} is determined.

On the other hand, when $n = 4m+2$ the series for g_n in (52) terminates, the term of highest degree in g_n being a multiple of η^{n+3}. In such cases it is not possible to arrange for the absence of exponentially

large terms in f_n by a suitable choice of α_{n-1}, and some other condition must be satisfied.

At each stage, in solving for f_n, the value of a_{n+2} is fixed by (42) in terms of such of the P_r as have occurred in the equations up to and including the equation for f_n and of such of the α_r, for $r \leqslant n-1$, as have not been determined by the conditions for the absence of exponentially large terms.

In order to proceed farther, we consider in some detail the equations for f_5 and f_6. The equation for f_5 may be written

$$f_5''' - \tfrac{1}{2}\eta^3 f_5'' + \tfrac{9}{2}\eta^2 f_5' - 8\eta f_5 = -14\alpha_1\alpha_4\eta^2 - \tfrac{8}{3}\alpha_1 P_1\left(\eta^3 + \frac{\eta^7}{30}\right) + \alpha_1^5 H_5(\eta), \tag{93}$$

where $H_5(\eta)$ is a function of η independent of α_1, α_4, and P_1, and the solution with a double zero at the origin is

$$f_5 = \alpha_5\eta^2 + 4\alpha_1\alpha_4(\eta - g_5) - \frac{\alpha_1 P_1}{45}\eta^6 + \alpha_1^5 k_5(\eta), \tag{94}$$

where $k_5(\eta)$ is a function of η independent of α_1, α_4, α_5, and P_1. The presence of exponentially large terms in the asymptotic expansion of f_5 can be avoided by a suitable choice of α_4, of the form

$$\alpha_4 = (\text{constant})\alpha_1^4. \tag{95}$$

The equation for f_6 is then found to be of the form

$$\begin{aligned} &f_6''' - \tfrac{1}{2}\eta^3 f_6'' + 5\eta^2 f_6' - 9\eta f_6 \\ &\quad = -16\alpha_1\alpha_5\eta^2 - \tfrac{26}{45}\alpha_1^2 P_1\eta^6 - \tfrac{8}{3}\alpha_2 P_1\left(\eta^3 + \frac{\eta^7}{20}\right) + \alpha_1^6 H_6(\eta), \end{aligned} \tag{96}$$

where $H_6(\eta)$ is independent of α_1, α_4, α_5, and P_1, and the solution with a double zero at the origin is

$$f_6 = \alpha_6\eta^2 + 4\alpha_1\alpha_5(\eta - g_6) - \frac{13\alpha_1^2 P_1}{11340}\eta^9 - \frac{\alpha_2 P_1}{45}\eta^6 + \alpha_1^6 k_6(\eta), \tag{97}$$

where $k_6(\eta)$ is independent of α_1, α_5, α_6, and P_1. From (52)

$$g_6 = \eta + \frac{\eta^5}{15} - \frac{\eta^9}{1260}. \tag{98}$$

Hence

$$f_6 = \alpha_6\eta^2 + \alpha_1\alpha_5\left(\frac{\eta^9}{315} - \frac{4\eta^5}{15}\right) - \frac{13\alpha_1^2 P_1}{11340}\eta^9 - \frac{\alpha_2 P_1}{45}\eta^6 + \alpha_1^6 k_6(\eta). \tag{99}$$

In order that the asymptotic expansion of f_6 should contain no exponentially large terms it is necessary that the asymptotic expansion of k_6 should contain no exponentially large terms, i.e. that the particular integral of

$$f_6''' - \tfrac{1}{2}\eta^3 f_6'' + 5\eta^2 f_6' - 9\eta f_6 = H_6(\eta) \tag{100}$$

with a double zero at the origin should contain no exponentially large

terms. This particular integral is indeterminate only to the extent of an additive multiple of η^2, and since the expansion of H_6 in ascending powers of η begins with a multiple of η^2, we may consider the condition to refer to the particular integral of which the expansion in ascending powers of η begins with a multiple of η^5. The condition may be given a more definite form. Put

$$f_6 = \eta^2 y_6, \quad \text{and} \quad y_6' = z_6. \tag{101}$$

Then (100) becomes

$$\eta^2 z_6'' + (6\eta - \tfrac{1}{2}\eta^5)z_6' + (6+3\eta^4)z_6 = H_6(\eta). \tag{102}$$

A complementary function of (102) (corresponding with the complementary function $g_6(\eta)$ of the equation for f_6) is

$$u_6 = -\frac{1}{\eta^2} + \frac{\eta^2}{5} - \frac{\eta^6}{180}. \tag{103}$$

If we put

$$z_6 = u_6 v_6 \quad \text{and} \quad v_6' = w_6, \tag{104}$$

(102) becomes

$$\frac{d}{d\eta}[u_6^2 \eta^6 w_6 \exp(-\eta^4/8)] = H_6 u_6 \eta^4 \exp(-\eta^4/8). \tag{105}$$

The expansion of the right-hand side of (105) in ascending powers of η begins with a multiple of η^4. As explained, the expansion of the required particular integral for f_6 begins with a multiple of η^5, and hence the expansion of $u_6^2 \eta^6 w_6 \exp(-\eta^4/8)$ begins with a multiple of η^5. The required solution of (105) is therefore

$$u_6^2 \eta^6 w_6 \exp(-\eta^4/8) = \int_0^\eta H_6 u_6 \eta^4 \exp(-\eta^4/8)\, d\eta. \tag{106}$$

If the asymptotic expansion of f_6 contains no exponentially large terms, the asymptotic expansion of w_6 contains no exponentially large terms, and the left-hand side of (106) $\to 0$ when $\eta \to \infty$. Hence the required condition is

$$\int_0^\infty H_6 u_6 \eta^4 \exp(-\eta^4/8)\, d\eta = 0, \tag{107}$$

i.e.

$$\int_0^\infty H_6 \left(\eta^2 - \frac{\eta^6}{5} + \frac{\eta^{10}}{180}\right) \exp(-\eta^4/8)\, d\eta = 0. \tag{108}$$

The expression for H_6 is easily found from the expression for G_6 in (46), but I have not been able to evaluate analytically the integral on the right of (108). The matter has been further considered by Mr. C. W. Jones, who finds numerically that the integral condition is satisfied to the accuracy of his computations.

If we assume for the present that the condition (108) is satisfied, we may proceed to consider the equations for f_7, f_8, etc., in the same way as

we considered the equations for f_5 and f_6. From the condition for the absence of exponentially large terms in the asymptotic expansion of f_7, α_6 is determined in terms of α_1, α_5, and P_1. Similarly α_7 is determined from the equation for f_8 in terms of α_1, α_5, and P_1,† and so on until we come to the equation for f_{10}. In the right-hand side, G_{10}, of the equation for f_{10}, α_9 occurs only in the term $-24\alpha_1\alpha_9\eta^2$, and the corresponding part of the particular integral with a double zero at the origin is $4\alpha_1\alpha_9(\eta-g_{10})$. Since the series for g_{10} terminates, the term of highest degree being a multiple of η^{13}, this part of the particular integral contains no exponentially large terms, and α_9 will not be determined from the condition that the asymptotic expansion of f_{10} should contain no exponentially large terms. G_{10} also contains a multiple of $\alpha_5^2\eta^2$ and certain very complicated terms linear in α_5, in particular a term in $\alpha_1^5\alpha_5$. (It further contains in particular complicated terms that are multiples of α_1^{10} and $\alpha_1^6 P_1$.) The part of the particular integral that corresponds with the multiple of $\alpha_5^2\eta^2$ in G_{10} will be a multiple of $\alpha_5^2(\eta-g_{10})$, and will not be exponentially large at infinity. Unless, therefore, the part of the particular integral corresponding with those complicated terms in G_{10} which are linear in α_5 also fails to become exponentially large at infinity, α_5 will be determined from the condition that f_{10} as a whole should not be exponentially large, which condition it will thus always be possible to fulfil. In other words, it seems probable (though I have not proved it) that α_5 is determined from the equation for f_{10}, α_9 from the equation for f_{14}, and so on. If this is correct, then only α_1 remains undetermined among the α's, and therefore only a_4 among the a's. As explained in the introduction, a_4 is then probably determined by the condition $u_1 \to 1$ as $y_1 \to \infty$ at $x_1 = 0$, in which case the whole solution is determined at separation, and probably what we have considered is an asymptotic solution at and near separation, applying (in the non-dimensional form here considered) to all cases in which the P's are the same.

3. The solution upstream for large values of $y_1/x_1^{\frac{1}{4}}$

Our first aim in this section is to exhibit the *form* of the solution for large values of $y_1/x_1^{\frac{1}{4}}$. By carrying through the suggested calculation in some detail we also provide a check on part of the work in the preceding section.

† The right-hand side, G_8, of the equation for f_8 contains P_2 and a term, $-\frac{P_1^2}{3}\left(\eta^4+\frac{\eta^8}{15}\right)$, in P_1^2. The corresponding parts of the particular integral with a double zero at the origin are, however,

$$P_2\left(\frac{\eta^3}{6}-\frac{\eta^7}{315}+\frac{\eta^{11}}{31185}\right) \quad \text{and} \quad -P_1^2\left(\frac{\eta^7}{630}+\frac{\eta^{11}}{155925}\right)$$

and contain no exponentially large terms, so that the expression for α_7 will involve neither P_2 nor P_1^2.

According to the solution in the preceding section, ψ_1 is found in the form of the series (24), where

$$f_0 = \frac{\eta^3}{6}, \qquad f_1 = \alpha_1 \eta^2, \qquad f_2 = \alpha_2 \eta^2 - \frac{\alpha_1^2}{15}\eta^5, \tag{109}$$

α_2 is given by (75), and for large positive values of η

$$f_3 \sim A_3 \eta^6 + C_3 \eta^4 + E_3' \eta^2 \log \eta + E_3 \eta^2 + F_3 \eta + G_3 + K_3 \eta^{-2} + \ldots, \tag{110}$$

where

$$\left.\begin{aligned} &A_3 = -\frac{\pi}{60(\frac{1}{4}!)^2}\alpha_1^3, \qquad C_3 = -\tfrac{2}{3}\alpha_1^3, \qquad E_3' = \frac{\pi}{(\frac{1}{4}!)^2}\alpha_1^3, \\ &E_3 = \frac{\pi}{4(\frac{1}{4}!)^2}\alpha_1^3\left\{2\log 2 + \gamma - \tfrac{1}{2}\pi - 5 + \frac{\pi^2}{100(\frac{1}{4}!)^4}(35 - 8\,.\,2^{\frac{1}{2}})\right\}, \\ &F_3 = \frac{4\,.\,2^{\frac{1}{4}}\,.\,\pi^{\frac{3}{2}}}{5(\frac{1}{4}!)^3}\alpha_1^3, \qquad G_3 = -\tfrac{8}{3}\alpha_1^3, \qquad K_3 = \frac{\pi}{8(\frac{1}{4}!)^2}\alpha_1^3, \end{aligned}\right\} \tag{111}$$

and

$$f_4 \sim A_4 \eta^7 + C_4 \eta^5 + D_4 \eta^4 + E_4 \eta^3 + F_4 \eta^2 + G_4' \eta \log \eta + G_4 \eta + K_4 \eta^{-1} + \ldots, \tag{112}$$

where

$$\left.\begin{aligned} &A_4 = \left[\frac{4}{63} - \frac{\pi^2}{150(\frac{1}{4}!)^4}\right]\alpha_1^4 - \frac{P_1}{630}, \qquad C_4 = -\frac{\pi}{5(\frac{1}{4}!)^2}\alpha_1^4, \\ &D_4 = -\frac{2^{\frac{3}{4}}\,.\,\pi^{\frac{3}{2}}}{15(\frac{1}{4}!)^3}\alpha_1^4, \qquad E_4 = \frac{P_1}{6} - \left[\frac{16}{3} - \frac{7\pi^2}{10(\frac{1}{4}!)^4}\right]\alpha_1^4, \\ &F_4 = \alpha_4 - \frac{21\,.\,2^{\frac{3}{4}}\,.\,\pi^4}{400(\frac{1}{4}!)^7}\alpha_1^4 \quad (\alpha_4 \text{ has not been calculated}), \\ &G_4' = \frac{4\pi}{(\frac{1}{4}!)^2}\alpha_1^4, \qquad G_4 \frac{\pi}{(\frac{1}{4}!)^2}\alpha_1^4\left[2\log 2 + \gamma - \tfrac{1}{2}\pi - 3 + \frac{7\pi^2}{20(\frac{1}{4}!)^4}\right], \\ &K_4 = \frac{7\pi^2}{20(\frac{1}{4}!)^4}\alpha_1^4. \end{aligned}\right\} \tag{113}$$

Now substitute the expressions for f_0, f_1, f_2 and the asymptotic formulae for f_3 and f_4 into the series (24) for ψ_1, substitute for η from $2^{\frac{1}{2}}\xi\eta = y_1$, and rearrange in powers of ξ and $\log \xi$. The work is purely formal, and no justification is attempted. The result, which can hold only for large values of $y_1/x_1^{\frac{1}{2}}$, is

$$\begin{aligned} \psi_1 = \frac{y_1^3}{6} - \frac{\alpha_1^2}{30}y_1^5 + \frac{A_3}{2^{\frac{3}{2}}}y_1^6 + \frac{A_4}{4}y_1^7 + \ldots \\ &+ \xi^2(2^{\frac{1}{2}}\alpha_1 y_1^2 + 2^{-\frac{1}{2}}C_3 y_1^4 + \tfrac{1}{2}C_4 y_1^5 + \ldots) + \\ &+ \xi^3(2^{\frac{1}{2}}\alpha_2 y_1^2 + 2^{-\frac{1}{2}}D_4 y_1^4 + \ldots) + \\ &+ \xi^4\{[2^{\frac{1}{2}}E_3 - 2^{-\frac{1}{2}}E_3' \log 2]y_1^2 + 2^{\frac{1}{2}}E_3' y_1^2 \log y_1 + E_4 y_1^3 + \ldots\} + \\ &+ \xi^4 \log \xi(-2^{\frac{1}{2}}E_3' y_1^2 + \ldots) + \\ &+ \xi^5(2F_3 y_1 + 2^{\frac{1}{2}}F_4 y_1^2 + \ldots) + \ldots. \end{aligned} \tag{114}$$

We are therefore led to assume, as a form valid for sufficiently large values of $y_1/x_1^{\frac{1}{4}}$,

$$\psi_1 = \chi_0(y_1)+\xi^2\chi_2(y_1)+\xi^3\chi_3(y_1)+\xi^4\chi_4(y_1)+(\xi^4\log\xi)\bar{\chi}_4(y_1)+\xi^5\chi_5(y_1)+..., \tag{115}$$

and the initial terms of the expansions of the χ in powers of y_1 are found by comparing with (114). In fact, since the form (115) is not valid unless $y_1/x_1^{\frac{1}{4}}$ is sufficiently large, the boundary conditions at $y_1 = 0$ cannot be applied, and the solution found must be joined to the solution in the preceding section by using the first few terms of the expansions of the χ's as given by (114). On the other hand, the solution in this section should satisfy the condition $u_1 \to U_1$ as $y_1 \to \infty$.

From (115)

$$u_1 = \frac{\partial\psi_1}{\partial y_1} = \chi_0'+\xi^2\chi_2'+\xi^3\chi_3'+\xi^4\chi_4'+(\xi^4\log\xi)\bar{\chi}_4'+\xi^5\chi_5'+..., \tag{116}$$

$$v_1 = \frac{\partial\psi_1}{\partial x_1} = \frac{1}{4\xi^2}\{2\chi_2+3\xi\chi_3+\xi^2(4\chi_4+\bar{\chi}_4)+4(\xi^2\log\xi)\bar{\chi}_4+5\xi^3\chi_5+...\}. \tag{117}$$

We substitute in the reduced equation of motion (16) (with $\partial p_1/\partial x_1$ as in (17)), multiply by $4\xi^2$, equate coefficients of ξ^0, ξ, ξ^2, $\xi^2\log\xi$, and ξ^3, and obtain the following equations:

$$\chi_0''\chi_2-\chi_0'\chi_2' = 0, \tag{118}$$

$$\chi_0''\chi_3-\chi_0'\chi_3' = 0, \tag{119}$$

$$\chi_0''(4\chi_4+\bar{\chi}_4)-\chi_0'(4\chi_4'+\bar{\chi}_4') = 4+4\chi_0'''+2(\chi_2'^2-\chi_2\chi_2''), \tag{120}$$

$$\chi_0''\bar{\chi}_4-\chi_0'\bar{\chi}_4' = 0, \tag{121}$$

$$5(\chi_0''\chi_5-\chi_0'\chi_5') = 5\chi_2'\chi_3'-3\chi_2''\chi_3-2\chi_2\chi_3''. \tag{122}$$

Now
$$\chi_0' = \tfrac{1}{2}y_1^2-\tfrac{1}{6}\alpha_1^2y_1^4+\frac{3A_3}{2^{\frac{1}{2}}}y_1^5+\frac{7A_4}{4}y_1^6+..., \tag{123}$$

and is, of course, the value of u_1 at $\xi = 0$, namely,

$$\tfrac{1}{2}y_1^2+a_3y_1^3+a_4y_1^4+...,$$

as may be verified from the values of A_3, A_4, a_3, a_4, a_5, a_6.

The solution of (118) is

$$\chi_2 = (\text{constant})\chi_0'. \tag{124}$$

The constant multiplier is determined by making the coefficient of y_1^2 in the expansion of χ_2 equal to $2^{\frac{1}{2}}\alpha_1$. Hence

$$\chi_2 = 2^{\frac{3}{2}}\alpha_1\chi_0'. \tag{125}$$

(It may now be verified that the coefficients of y_1^4 and y_1^5 in the expansion of χ_2 are the same as in (114), and the values of C_3 and C_4 are thus checked.)

Similarly from (119) and (121) it follows that χ_3 and $\bar{\chi}_4$ are constant

multiples of χ_0', and the multipliers are determined from the coefficients of y_1^2 by comparison with (114). In these multipliers the values of α_2 and E_3' are substituted from (75) and (111), and it is found that

$$\chi_3 = \frac{2^{\frac{7}{4}}.\pi^{\frac{3}{2}}}{5(\frac{1}{4}!)^3}\alpha_1^2\chi_0', \tag{126}$$

$$\bar{\chi}_4 = -\frac{2^{\frac{3}{2}}\pi}{(\frac{1}{4}!)^2}\alpha_1^3\chi_0'. \tag{127}$$

(The coefficient of y_1^4 in the expansion of χ_3 may now be verified, and the value of D_4 so checked.)

In (120) the terms in $\bar{\chi}_4$ may be omitted, since they cancel by (121), and the value of χ_2 may be substituted from (125). The equation for χ_4 is thus seen to be

$$\frac{d}{dy_1}\left(\frac{\chi_4}{\chi_0'}\right) = \chi_0'^{-2}\{1-\chi_0'''-4\alpha_1^2(\chi_0''^2-\chi_0'\chi_0''')\}. \tag{128}$$

The expression on the right of (128) may be expanded in a series of ascending powers of y_1; the first terms are

$$-\frac{120.2^{\frac{1}{2}}.A_3}{y_1}+(\tfrac{8}{3}\alpha_1^4-210A_4)+\dots. \tag{129}$$

Hence the solution of (128) is

$$\chi_4 = \chi_0'\int\limits_0^{y_1}\left\{\frac{1-\chi_0'''-4\alpha_1^2(\chi_0''^2-\chi_0'\chi_0''')}{\chi_0'^2}+\frac{120.2^{\frac{1}{2}}.A_3}{y_1}\right\}dy_1-$$
$$-120.2^{\frac{1}{2}}.A_3\chi_0'\log y_1+(\text{constant})\chi_0'. \tag{130}$$

The constant multiplier of χ_0' is found from the coefficient of y_1^2 to be equal to

$$2^{\frac{3}{2}}E_3-2^{\frac{1}{2}}E_3'\log 2 = \frac{2^{\frac{1}{2}}\pi}{2(\frac{1}{4}!)^2}\alpha_1^3\left\{\gamma-\tfrac{1}{2}\pi-5+\frac{\pi^2}{100(\frac{1}{4}!)^4}(35-8.2^{\frac{1}{2}})\right\}. \tag{131}$$

(The coefficients of $y_1^2\log y_1$ and y_1^3 may now be verified, and the values of E_3' and E_4 so checked.)

When we substitute for χ_2 and χ_3 from (125) and (126), the equation (122) for χ_5 reduces to

$$\chi_0''\chi_5-\chi_0'\chi_5' = \frac{8.2^{\frac{1}{4}}.\pi^{\frac{3}{2}}}{5(\frac{1}{4}!)^3}\alpha_1^3(\chi_0''^2-\chi_0'\chi_0''') \tag{132}$$

with the solution

$$\chi_5 = \frac{8.2^{\frac{1}{4}}.\pi^{\frac{3}{2}}}{5(\frac{1}{4}!)^3}\alpha_1^3\chi_0''+(\text{constant})\chi_0'. \tag{133}$$

From the coefficient of y_1^2 the constant multiplier is seen to be $2^{\frac{3}{2}}F_4$, but it cannot be fully determined from results already found, since the expression for F_4 in (113) contains α_4, which has not been calculated.

(The coefficient of y_1 in the expansion of χ_5 is determined without a knowledge of the constant multiplier, and is easily seen to be $2F_3$, as in (114). The value of F_3 is thus checked.)

It remains to consider the condition that the velocity should pass smoothly over into the velocity of the main stream. As explained, we must suppose that this condition is satisfied at $x_1 = 0$; as far as our formulae go, u_1 is given by (116), so if the condition is satisfied at $x_1 = 0$, $\chi'_0 \to 1$, and the second and higher derivatives $\to 0$, as $y_1 \to \infty$. From (125), (126), (127), and (133) it follows at once that χ'_2, χ'_3, χ'_4, χ'_5 and their derivatives all $\to 0$ as $y_1 \to \infty$. As regards χ_4, we suppose that, as $y_1 \to \infty$, $\chi''_0 \to 0$ more rapidly than y_1^{-1}; it may then be proved that as $y_1 \to \infty$, χ_4 is asymptotically equal to a multiple of y_1, plus a constant, plus terms which $\to 0$, so that $\chi_4 \to \infty$ as $y_1 \to \infty$; but $\chi''_0 \chi_4 \to 0$, and we may then show from (118) that $\chi'_4 \to 1$ and the second and higher derivatives $\to 0$, as $y_1 \to \infty$. We thus check that, as far as (116) goes, $u_1 \to U_1$ (since $\xi^4 = x_1$ and $U_1 = 1$, $dU_1/dx_1 = 1$ at $x_1 = 0$), while $\partial u_1/\partial y_1$ and higher derivatives $\to 0$.

4. Discussion of the solution upstream

The solution in the preceding section holds only for large values of η. The solution in § 2 applies for small or moderate values of η; it is valid also for large values of η (the asymptotic expansions of f_3 and f_4 being used), but is then useful only for small values of y_1, since in such a case it is essentially equivalent to the solution in § 3 with the χ's expanded in series and with only a few terms in each expansion known. The results that can be obtained from § 2 when η is large may therefore be more advantageously obtained from § 3 after we have obtained the formulae for the coefficients a_n in the expansion, in powers of y_1, of the value of u_1 at $x_1 = 0$. (It should also be remembered that the asymptotic formulae of § 2 were necessary for the completion of the solution in § 3.)

Thus the formulae in §§ 2 and 3 are useful in different regions. In particular, the formulae of § 2 will be useful for studying the values at $y_1 = 0$, $x_1 \neq 0$ of the derivatives of u_1 with respect to y_1, etc., whereas the formulae of § 3 will be useful for studying the nature of the solution at $x_1 = 0$, $y_1 \neq 0$. The formulae of § 2 show that $\lim_{x_1 \to 0} (\partial^n u_1/\partial y_1^n)_{y_1=0} \neq n!\, a_n$ for $n = 5$ and 6. For any non-zero value of y_1, however, $\partial^n u_1/\partial y_1^n$ is continuous at $x_1 = 0$. On the other hand, the formulae of § 3 show that v_1 (and $\partial u_1/\partial x_1$) are infinite at $x_1 = 0$.

Because of the singularity at $x_1 = 0$ the usual assumptions of boundary-layer theory are invalid at $x_1 = 0$ and in the immediate neighbourhood. Nevertheless, the mathematical result that v_1 is infinite may be taken to indicate that large cross-velocities are to be expected at separation;

otherwise the assumptions of boundary-layer theory would not break down.

5. The solution downstream

In considering the motion downstream from separation we reduce the equations of motion and continuity to non-dimensional form by the same substitutions as before, except that in place of x_1 we use

$$x_1' = -x_1 = (x - x_s)/l, \tag{134}$$

so that x_1' is positive downstream. The governing equations are obtained by replacing x_1 by $-x_1'$ in (16) and (17), and in place of (23) and (24) we write

$$\xi' = x_1'^{\frac{1}{4}}, \qquad \eta' = y_1/2^{\frac{1}{2}}x'^{\frac{1}{4}}, \tag{135}$$

$$\psi_1 = 2^{\frac{3}{2}}\xi'^3[F_0(\eta') + \xi' F_1(\eta') + \xi'^2 F_2(\eta') + \ldots]. \tag{136}$$

We obtain differential equations for the F in the same way as before; moreover, the boundary conditions are the same, for the conditions $\psi_1 = 0$ and $u_1 = 0$ at $y_1 = 0$ lead to

$$F_r(0) = F_r'(0) = 0 \quad (r = 0, 1, 2, \ldots) \tag{137}$$

and the condition that $\lim_{\xi' \to 0} u_1$ is given by (21) leads to

$$\lim_{\eta' \to \infty} \frac{F_r'}{\eta'^{r+2}} = 2^{\frac{1}{2}r} a_{r+2} \quad (r = 0, 1, 2, \ldots). \tag{138}$$

The equation for F_0 is

$$F_0''' + 3F_0 F_0'' - 2F_0'^2 = 1, \tag{139}$$

and, since $a_2 = \frac{1}{2}$, the solution is the same as for f_0, namely,

$$F_0 = \eta'^3/6. \tag{140}$$

The equation for F_1 is

$$F_1''' + \tfrac{1}{2}\eta'^3 F_1'' - \tfrac{5}{2}\eta'^2 F_1' + 4\eta' F_1 = 0, \tag{141}$$

and, since $a_3 = 0$, the solution is

$$F_1 = \beta_1 \eta'^2, \tag{142}$$

where β_1 is a constant. The equation for F_2 now becomes

$$F_2''' + \tfrac{1}{2}\eta'^3 F_2'' - 3\eta'^2 F_2' + 5\eta' F_2 = 4\beta_1^2 \eta'^2, \tag{143}$$

and the general solution for F_2 with a double zero at the origin is

$$F_2 = \beta_2 \eta'^2 + \tfrac{1}{15}\beta_1^2 \eta'^5, \tag{144}$$

where β_2 is a constant. Hence, from (138),

$$a_4 = \tfrac{1}{6}\beta_1^2. \tag{145}$$

But, from (50), a_4 is zero (in which case α_1 is zero) or negative. If a_4

is negative, there is no real solution downstream of separation. Hence there is no real solution downstream of separation unless $a_4 = 0$.

6. The special case $a_4 = 0$. The solution without singularities

If we return to § 2, and consider the motion upstream of separation when $\alpha_1 = 0$, the equations for the f are easily integrated, and we may show that $\alpha_2 = \alpha_4 = \alpha_5 = \alpha_6 = \alpha_8 = \alpha_9 =$ etc. $= 0$; the whole solution is free from singularities and may be verified by expanding ψ_1 in a double power series in x_1 and y_1. Moreover, since the solution is free from singularities it will hold also downstream. It is not possible, however, by the methods used, to consider if there are pressure distributions for which $u_1 \to 1$ as $y_1 \to \infty$ at $x_1 = 0$.

The most interesting special case of this solution is that in which $\alpha_3 = \alpha_7 = \ldots = 0$, when it reduces to one that is easily found independently, namely (with $x_1' = -x_1$),

$$\psi_1 = \frac{1}{2}\frac{\partial p_1}{\partial x_1'}\frac{y_1^3}{3} + \frac{1}{360}\frac{\partial^2 p_1}{\partial x_1'^2}\frac{y_1^7}{7} + \frac{5\left(\dfrac{\partial p_1}{\partial x_1'}\right)^2 \dfrac{\partial^3 p_1}{\partial x_1'^3} - 2\dfrac{\partial p_1}{\partial x_1'}\left(\dfrac{\partial^2 p_1}{\partial x_1'^2}\right)^2}{453600}\frac{y_1^{11}}{11} + \ldots, \tag{146}$$

so the expansion of u_1 in powers of y_1 contains only multiples of y_1^{4m+2}. This is a solution in which $(\partial u/\partial y)_{y=0} = 0$ for all x, but in order that it should be valid it is necessary that $\partial p_1/\partial x_1'$ should be chosen so that $u_1 \to 1$ when $y_1 \to \infty$ at $x_1' = 0$. Included in this solution is that special case of the solution discovered by Falkner and Skan (4) for which $(\partial u/\partial y)_{y=0} = 0$ at all values of x. In the case considered by Falkner and Skan the velocity distributions at different values of x are similar; if more general solutions of the type shown in (146) exist, the velocity distributions at different values of x will not be similar in the general case.

It is a fairly straightforward matter to check that the known solution for $U = cx^m$, when m has the appropriate value, agrees with (146) as far as that equation goes; but the value of m is determined from the condition $u \to U$ as $y \to \infty$ and cannot be found by the methods used here. Since the velocity distributions at different values of x are similar, the appropriate value of m may be found from the solutions of an ordinary differential equation, and has been so found by Hartree (**5**).† No such method is available in the general case. Meanwhile the formulae of Falkner and Skan, when $(\partial u/\partial y)_{y=0} = 0$, have been fitted as a very special case into the formulae of this section, so far as those formulae go.

† For negative values of m the solution of the equation with the conditions $\psi = 0$, $u = 0$ at $y = 0$ and $u/U \to 1$ as $y \to \infty$ is not unique, but may be made unique by requiring that $1-u/U$ shall be positive and shall $\to 0$ exponentially as $y \to \infty$.

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NOTE ON THE VELOCITY AND TEMPERATURE DISTRIBUTIONS ATTAINED WITH SUCTION ON A FLAT PLATE OF INFINITE EXTENT IN COMPRESSIBLE FLOW†

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SUMMARY

The problem considered by Griffith and Meredith (**1**) for incompressible flow is here considered for compressible flow of a perfect gas with constant specific heats, it being assumed that there is no heat transfer by conduction at the plate. Essentially, the method consists in establishing a correspondence between the velocity and temperature profiles for incompressible flow and those for compressible flow, the lateral ordinates being scaled by factors which are functions of the ordinates and of Mach number.

The results of calculations covering a range of Mach numbers up to 5·0 are shown in Figs. 1 and 2.

1. Notation

- x distance measured parallel to the plate in direction of main stream upstream of plate
- y distance measured normal to the plate from the surface of the plate
- u velocity component in x-direction
- v velocity component in y-direction
- ρ density
- T temperature
- μ coefficient of viscosity
- k thermal conductivity
- c_v specific heat at constant volume (assumed constant)
- c_p specific heat at constant pressure (assumed constant)
- σ $\mu c_p/k$ (Prandtl number, assumed constant)
- J mechanical equivalent of heat
- γ c_p/c_v (assumed constant)
- i $Jc_p T$ (enthalpy)
- τ $\mu\dfrac{du}{dy}$ (shear stress)

suffix 1 refers to quantities measured at large normal distances from the plate ($y \to \infty$)

† This paper is substantially Report No. 8 of the College of Aeronautics.

suffix w refers to quantities measured at the plate

ω defined by $\left(\frac{\mu}{\mu_1}\right) = \left(\frac{T}{T_1}\right)^{\omega}$

η $\frac{yv_1}{\nu_1}$

ζ $\int_0^{\eta} \left(\frac{\mu_1}{\mu}\right) d\eta$

M_1 $\frac{u_1}{a_1} = \sqrt{\left\{\frac{u_1^2}{(\gamma-1)Jc_p T}\right\}}$ (a_1 is the speed of sound in the main stream)

θ i/i_1

b $(\gamma-1)M_1^2$.

2. Introduction

The solution due to Griffith and Meredith (**1**) of the velocity distribution attained with suction on a flat plate of infinite extent in incompressible flow is of special interest, since it is a solution of the general equations of motion and does not depend on the usual assumptions of boundary layer theory. The corresponding problem for compressible flow is by no means as simple in its most general form. However, if the usual assumptions of boundary layer theory are made, it permits of an exact solution which is easily obtained. In the following it is assumed that the gas is perfect and that the specific heats and Prandtl number are constant.

3. Analysis

The equation of motion in the boundary layer of a flat plate at zero incidence in steady compressible flow is

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right). \tag{1}$$

The equation of continuity is

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v) = 0. \tag{2}$$

The energy equation is

$$Jc_p u\frac{\partial T}{\partial x}+Jc_p v\frac{\partial T}{\partial y} = \frac{J}{\rho}\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\mu}{\rho}\left(\frac{\partial u}{\partial y}\right)^2. \tag{3}$$

We are interested in the problem of the final velocity and temperature profiles far downstream from the leading edge of the plate when all quantities are independent of x.

Hence, the above equations become

$$\rho v \frac{du}{dy} = \frac{d}{dy}\left(\mu \frac{du}{dy}\right), \tag{4}$$

$$\rho v = \text{const.} = \rho_1 v_1, \tag{5}$$

$$\rho v \frac{di}{dy} = \frac{d}{dy}\left(\frac{\mu}{\sigma}\frac{di}{dy}\right) + \mu\left(\frac{du}{dy}\right)^2, \tag{6}$$

where $i = Jc_p T$ and $\sigma = \frac{\mu c_p}{k}$ (Prandtl's number, assumed constant).

The gas equation leads to

$$\frac{\rho}{\rho_1} = \frac{T_1}{T} = \frac{i_1}{i}. \tag{7}$$

It will be assumed that the variation of μ with T is given by

$$\frac{\mu}{\mu_1} = \left(\frac{T}{T_1}\right)^\omega = \left(\frac{i}{i_1}\right)^\omega, \tag{8}$$

where $\omega = \text{const.}$ For air at normal temperatures ω is about 0·76, but it increases slightly with T_1.

The boundary conditions are

$$u = u_1, \quad \rho = \rho_1, \quad v = v_1, \quad i = i_1, \quad \frac{du}{dy} = 0 \quad \text{at} \quad y = \infty,$$

$$u = 0, \quad \frac{di}{dy} = 0 \quad \text{at} \quad y = 0,$$

if no heat transfer by conduction is assumed to occur at the plate.

If in equation (6) we change the independent variable from y to u, writing $\tau(u) = \mu \frac{du}{dy}$, $i = i(u)$ and eliminate ρv by means of equation (4), we obtain

$$(1-\sigma)\frac{d\tau}{du}\frac{di}{du} + \tau\left(\frac{d^2 i}{du^2} + \sigma\right) = 0. \tag{9}$$

From (4) and (5), $$\frac{d\tau}{du} = \rho v = \rho_1 v_1,$$

and hence $$\tau = \rho_1 v_1 u + C_1,$$

where C_1 is a constant.

If we write τ_w for the value of τ at the wall, $C_1 = \tau_w$. Further, since $\tau = 0$, when $u = u_1$, $C_1 = -\rho_1 v_1 u_1$. Therefore

$$\left.\begin{aligned} \tau &= -\rho_1 v_1 (u_1 - u) \\ &= \rho_1 v_1 u + \tau_w \end{aligned}\right\}. \tag{10}$$

Equation (9) can then be written

$$(1-\sigma)\frac{di}{du}-(u_1-u)\left(\frac{d^2i}{du^2}+\sigma\right)=0, \tag{11}$$

with the solution

$$i_1-i=\frac{\sigma u_1^2}{2(2-\sigma)}\left\{\left(1-\frac{u}{u_1}\right)^2-\frac{2}{\sigma}\left(1-\frac{u}{u_1}\right)^\sigma\right\} \tag{12}$$

satisfying the conditions $i = i_1$, when $u = u_1$, and

$$\frac{di}{du}=0, \quad \text{when} \quad u=0.$$

At the wall, where $u = 0$,

$$i_w = i_1+\frac{u_1^2}{2}, \tag{13}$$

and hence the total energy at the wall differs from that in the main stream only by the quantity $(v_w^2-v_1^2)$.

From (10) we have, since $\tau = \mu\dfrac{du}{dy}$,

$$\frac{u}{u_1}=1-\exp\left(\rho_1 v_1\int\limits_0^y \frac{dy}{\mu}\right). \tag{14}$$

Let
$$\eta = -\frac{v_1 y}{\nu_1},$$

and let
$$-d\zeta = d\eta\frac{\mu_1}{\mu}$$
$$= d\eta\left(\frac{T_1}{T}\right)^\omega = d\eta\left(\frac{i_1}{i}\right)^\omega, \tag{15}$$

with $\zeta = 0$, when $\eta = 0$.

Then, from (14),
$$\frac{u}{u_1}=1-\exp(\zeta), \tag{16}$$

and from (12),
$$i_1-i=\frac{\sigma u_1^2}{2(2-\sigma)}\left\{\exp(2\zeta)-\frac{2}{\sigma}\exp(\sigma\zeta)\right\}. \tag{17}$$

Writing $\theta = i/i_1$, $b = (\gamma-1)M_1^2$, then

$$\left.\begin{aligned}\frac{\theta-1}{\frac{1}{2}b} &= \frac{\sigma}{2-\sigma}F(\zeta),\\ \text{where}\quad F(\zeta) &= \frac{2}{\sigma}\exp(\sigma\zeta)-\exp(2\zeta)\end{aligned}\right\}. \tag{18}$$

From (16) and (18) we can express u/u_1 and $(\theta-1)/\frac{1}{2}b$ as functions of ζ only, independent of Mach number. To derive the actual velocity and temperature distributions for any given Mach number we need to evaluate the relation between ζ and η (or y) given by (15).

From (15), $$-\eta = \int\limits_0^\zeta \left(\frac{i}{i_1}\right)^\omega d\zeta = \int\limits_0^\zeta \left\{1+\frac{b\sigma}{2(2-\sigma)}F(\zeta)\right\}^\omega d\zeta. \tag{19}$$

In general, the integral on the right-hand side of (19) must be evaluated either numerically or graphically, giving η as a function of ζ and M_1. Since v_1 is negative, only negative values of ζ need be considered and it will be found that values of $|\zeta|$ greater than 10 may be ignored. Having determined η (or y) for a comprehensive range of values of ζ and M_1 we can then, for each Mach number, replot u/u_1 and $(\theta-1)/\frac{1}{2}b$ as functions of η, using the basic (or incompressible) profiles given by (16) and (18).†

For the special case $\omega = 1\cdot0$, (19) can be integrated outright to give

$$-\eta = \zeta+\frac{b}{2}\frac{\sigma}{2-\sigma}\left\{\frac{2}{\sigma^2}\exp(\sigma\zeta)-\frac{\exp(2\zeta)}{2}-\frac{2}{\sigma^2}+\frac{1}{2}\right\}. \tag{20}$$

4. Calculations and results

The velocity and temperature distributions have been calculated for $\omega = 0\cdot76$ and $M_1 = 0$, $1\cdot0$, $2\cdot0$, $3\cdot0$, $4\cdot0$, and $5\cdot0$, σ being taken as $0\cdot72$.

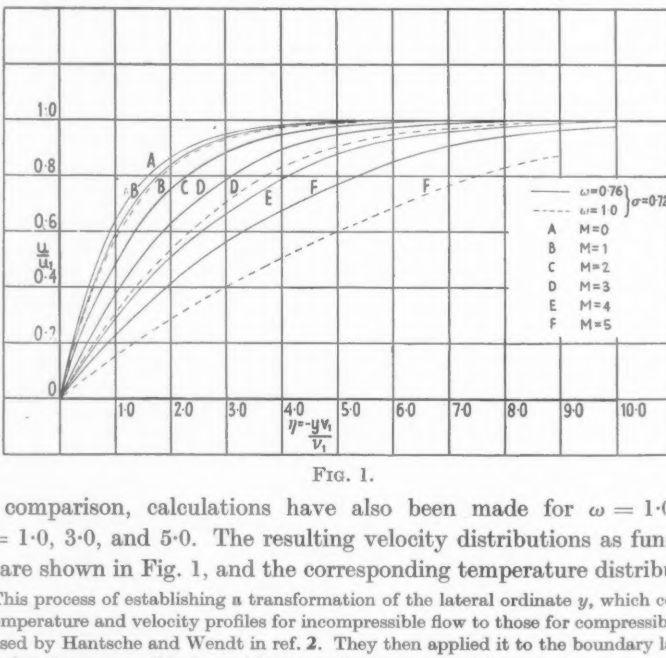

FIG. 1.

For comparison, calculations have also been made for $\omega = 1\cdot0$ and $M_1 = 1\cdot0$, $3\cdot0$, and $5\cdot0$. The resulting velocity distributions as functions of η are shown in Fig. 1, and the corresponding temperature distributions

† This process of establishing a transformation of the lateral ordinate y, which converts the temperature and velocity profiles for incompressible flow to those for compressible flow, was used by Hantsche and Wendt in ref. **2**. They then applied it to the boundary layer on a flat plate in compressible flow without suction for the special case where $\omega = 1\cdot0$. However, it seems capable of much wider application, and it is hoped to use it for more general problems of the boundary layer on a finite flat plate both with and without suction in compressible flow.

are shown in Fig. 2. It will be noted that there is a thickening of the velocity and temperature boundary layer with increase of Mach number, and this process is enhanced by an increase of ω.

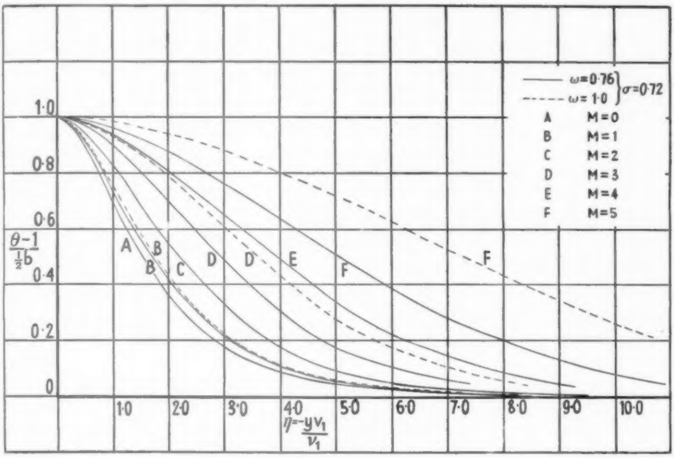

Fig. 2.

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SUPERSONIC FLOW PAST SLENDER POINTED BODIES OF REVOLUTION AT YAW

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SUMMARY

The effect of a slender body of revolution of thickness t and arbitrary shape, held with its axis at an angle ψ to a uniform supersonic stream, is considered. By use of the exact equations of isentropic motion an expression for the pressure coefficient is obtained accurate enough to give the lift and moment coefficients as sums of terms of orders ψ, ψ^3, ψt^2, and $\psi t^2 \log t^{-1}$, and the drag coefficient as a sum of terms of orders t^2, $t^2 \log t^{-1}$, and ψ^2, terms of smaller order (only) being neglected in each case.

1. Introduction

APPROXIMATIONS to the lift and moment on a slender pointed body of revolution whose axis is inclined at a small angle ψ to a uniform supersonic stream were obtained by Tsien (**1**), by solving the 'linearized equation of motion' both approximately and accurately. For a shell his first approximation can be written $C_L = 2\psi$, and this is correct if terms of order ψ^3 and ψt^2 (t the maximum thickness) are neglected. In this paper, by use of the exact equations of isentropic flow, additional terms in C_L of orders ψ^3 and ψt^2 are obtained, so that the new expression is correct if terms in ψ^5, $\psi^3 t^2$, and ψt^4 are neglected. The effect of the change of entropy through the shock-wave will only be felt in these latter orders of small quantities (this remark will be supported below, § 5): hence it is legitimate to use the equations of isentropic flow for our purpose. In addition to the lift, the pressure distribution on the body and the moment of the forces about the nose are found, as well as the extra drag due to yaw.

It is found that to obtain better approximations than $C_L = 2\psi$ the use of the linearized equation is inadequate. In a worked example the variation of the lift coefficient with the thickness of the body and with the Mach number is compared with that given by the exact solution of the linearized equation.

The principal application will be to shells and other supersonic projectiles. In this paper the meridian section of the body is assumed to have continuous slope, in fact to be an analytic arc; a partial extension to curves made up of a number of analytic arcs is given in a later paper (**2**).

2. Mathematical formulation of the problem

In cylindrical polar coordinates (r, θ, x) a uniform stream making an angle ψ with the axis has velocity potential $Ux\cos\psi + U\sin\psi . r\cos\theta$. Here θ is measured from the line which is the projection of the velocity vector on to a plane perpendicular to the axis. If a body of revolution is held fixed on the axis, the resulting flow, which as an approximation when the body is slender we assume to be irrotational (in the region ahead of the flat base if the body has one), will have a velocity potential ϕ which we will assume expansible in powers of ψ as follows:

$$\phi = Ux\cos\psi + U\sin\psi . r\cos\theta + \phi_0(x,r) + \psi\phi_1(x,r)\cos\theta + \\ + \psi^2\phi_2(x,r,\theta) + \psi^3\phi_3(x,r,\theta) + \dots . \quad (1)$$

It will be found that with a potential of this form the equation of motion and boundary conditions can be satisfied.

Now we have for the square of the velocity (using the suffix-derivative notation)

$$\begin{aligned} q^2 &= \phi_x^2 + \phi_r^2 + r^{-2}\phi_\theta^2 \\ &= (U\cos\psi + \phi_{0x} + \psi\phi_{1x}\cos\theta + \psi^2\phi_{2x} + \psi^3\phi_{3x} + \dots)^2 + \\ &\quad + (U\sin\psi\cos\theta + \phi_{0r} + \psi\phi_{1r}\cos\theta + \psi^2\phi_{2r} + \psi^3\phi_{3r} + \dots)^2 + \\ &\quad + r^{-2}(-U\sin\psi . r\sin\theta - \psi\phi_1\sin\theta + \psi^2\phi_{2\theta} + \dots)^2 \\ &= U^2 + Q_0 + \psi Q_1\cos\theta + \psi^2 Q_2 + \psi^3 Q_3 + \dots, \end{aligned} \quad (2)$$

$$\left.\begin{aligned} \text{where } Q_0 &= 2U\phi_{0x} + \phi_{0x}^2 + \phi_{0r}^2, \\ Q_1 &= 2(U+\phi_{0x})\phi_{1x} + 2(U+\phi_{1r})\phi_{0r}, \\ Q_2 &= -U\phi_{0x} + 2(U+\phi_{0x})\phi_{2x} + \phi_{1x}^2\cos^2\theta + 2\phi_{0r}\phi_{2r} + \\ &\quad + (2U\phi_{1r} + \phi_{1r}^2)\cos^2\theta + (2Ur^{-1}\phi_1 + r^{-2}\phi_1^2)\sin^2\theta, \\ Q_3 &= 2(U+\phi_{0x})\phi_{3x} + 2\phi_{1x}\cos\theta(\phi_{2x} - \tfrac{1}{2}U) + 2\phi_{0r}(\phi_{3r} - \tfrac{1}{6}U\cos\theta) + \\ &\quad + 2(U+\phi_{1r})\phi_{2r}\cos\theta - 2(Ur+\phi_1)\phi_{2\theta}r^{-2}\sin\theta. \end{aligned}\right\} \quad (3)$$

The equation of motion is $2a^2\nabla^2\phi = (\nabla q^2)(\nabla\phi)$, where

$$a^2 = a_0^2 + \tfrac{1}{2}(\gamma-1)(U^2 - q^2);$$

this can be written (if ψ^4 be neglected) in the form

$$\begin{aligned} &2\{a_0^2 - \tfrac{1}{2}(\gamma-1)(Q_0 + \psi Q_1\cos\theta + \psi^2 Q_2 + \psi^3 Q_3)\} \times \\ &\qquad \times \{\nabla^2\phi_0 + \psi(\nabla^2\phi_1 - r^{-2}\phi_1)\cos\theta + \psi^2\nabla^2\phi_2 + \psi^3\nabla^2\phi_3)\} \\ &= (U\sin\psi\cos\theta + \phi_{0r} + \psi\phi_{1r}\cos\theta + \psi^2\phi_{2r} + \psi^3\phi_{3r}) \times \\ &\qquad \times (Q_{0r} + \psi Q_{1r}\cos\theta + \psi^2 Q_{2r} + \psi^3 Q_{3r}) + \\ &\quad + (U\cos\psi + \phi_{0x} + \psi\phi_{1x}\cos\theta + \psi^2\phi_{2x} + \psi^3\phi_{3x}) \times \\ &\qquad \times (Q_{0x} + \psi Q_{1x}\cos\theta + \psi^2 Q_{2x} + \psi^3 Q_{3x}) + \\ &\quad + r^{-2}(-U\sin\psi . r\sin\theta - \psi\phi_1\sin\theta + \psi^2\phi_{2\theta})(-\psi Q_1\sin\theta + \psi^2 Q_{2\theta}). \end{aligned} \quad (4)$$

Equating coefficients we deduce that

$$2\{a_0^2-\tfrac{1}{2}(\gamma-1)Q_0\}\nabla^2\phi_0 = \phi_{0r}\,Q_{0r}+(U+\phi_{0x})Q_{0x}, \tag{5}$$

$$\begin{aligned}2\{a_0^2-\tfrac{1}{2}(\gamma-1)Q_0\}(\nabla^2\phi_1-r^{-2}\phi_1)-(\gamma-1)Q_1\nabla^2\phi_0 \\ = (U+\phi_{1r})Q_{0r}+\phi_{0r}\,Q_{1r}+(U+\phi_{0x})Q_{1x}+\phi_{1x}\,Q_{0x},\end{aligned} \tag{6}$$

$$\begin{aligned}&2\{a_0^2-\tfrac{1}{2}(\gamma-1)Q_0\}\nabla^2\phi_2-(\gamma-1)Q_1(\nabla^2\phi_1-r^{-2}\phi_1)\cos^2\theta-(\gamma-1)Q_2\nabla^2\phi_0 \\ &\quad= \phi_{0r}\,Q_{2r}+(U+\phi_{1r})Q_{1r}\cos^2\theta+\phi_{2r}\,Q_{0r}+(U+\phi_{0x})Q_{2x}+\phi_{1x}\,Q_{1x}\cos^2\theta+ \\ &\qquad+(\phi_{2x}-\tfrac{1}{2}U)Q_{0x}+r^{-2}(Ur+\phi_1)Q_1\sin^2\theta,\end{aligned} \tag{7}$$

$$\begin{aligned}&2\{a_0^2-\tfrac{1}{2}(\gamma-1)\phi_0\}\nabla^2\phi_3-(\gamma-1)Q_1\cos\theta\,\nabla^2\phi_2-(\gamma-1)Q_2\cos\theta(\nabla^2\phi_1-r^{-2}\phi_1)- \\ &\qquad-(\gamma-1)Q_3\nabla^2\phi_0 \\ &= \phi_{0r}\,Q_{3r}+(U+\phi_{1r})Q_{2r}\cos\theta+\phi_{2r}\,Q_{1r}\cos\theta+(\phi_{3r}-\tfrac{1}{6}U\cos\theta)Q_{0r}+ \\ &\qquad+(U+\phi_{0x})Q_{3x}+\phi_{1x}\,Q_{2x}\cos\theta+(\phi_{2x}-\tfrac{1}{2}U)Q_{1x}\cos\theta+\phi_{3x}\,Q_{0x}- \\ &\qquad-Q_1\,\phi_{2\theta}\,r^{-2}\sin\theta-(Ur+\phi_1)r^{-2}Q_{2\theta}\sin\theta.\end{aligned} \tag{8}$$

The boundary condition is that $\phi_r/\phi_x = R'(x)$ when $r = R(x)$ if this latter is the equation of the meridian section. The condition becomes

$$\begin{aligned}&U\sin\psi\cos\theta+\phi_{0r}+\psi\phi_{1r}\cos\theta+\psi^2\phi_{2r}+\psi^3\phi_{3r}+\ldots \\ &\qquad= R'(x)(U\cos\psi+\phi_{0x}+\psi\phi_{1x}\cos\theta+\psi^2\phi_{2x}+\psi^3\phi_{3x}+\ldots).\end{aligned}$$

Hence the boundary conditions appropriate to equations (5), (6), (7), and (8) are

$$\phi_{0r} = R'(x)(U+\phi_{0x}), \tag{5a}$$

$$U+\phi_{1r} = R'(x)\phi_{1x}, \tag{6a}$$

$$\phi_{2r} = R'(x)(-\tfrac{1}{2}U+\phi_{2x}), \tag{7a}$$

$$-\tfrac{1}{6}U\cos\theta+\phi_{3r} = R'(x)\phi_{3x}, \tag{8a}$$

all on $r = R(x)$. There is also a condition, which we call 'C', that, at infinity, the potentials ϕ_i shall represent disturbances travelling in such a way that x increases with r.

When the ϕ_i have been determined the pressure is obtained by the relation

$$\begin{aligned}\frac{p}{p_0} = \left(\frac{a^2}{a_0^2}\right)^{\gamma/(\gamma-1)} &= \left(1+\frac{\gamma-1}{2a_0^2}(U^2-q^2)\right)^{\gamma/(\gamma-1)} \\ &= 1+\frac{\gamma}{2a_0^2}(U^2-q^2)+\frac{\gamma}{8a_0^4}(U^2-q^2)^2+\ldots,\end{aligned} \tag{9}$$

or

$$\begin{aligned}C_p = \frac{p-p_0}{\frac{1}{2}\rho_0 U^2} &= \frac{U^2-q^2}{U^2}+\tfrac{1}{4}M^2\left(\frac{U^2-q^2}{U^2}\right)^2+\ldots \\ &= -\frac{Q_0}{U^2}-\frac{\psi Q_1\cos\theta}{U^2}-\frac{\psi^2 Q_2}{U^2}-\frac{\psi^3 Q_3}{U^2}+\tfrac{1}{4}M^2U^{-4}(Q_0+\psi Q_1\cos\theta+\psi^2Q_2+\psi^3Q_3)^2,\end{aligned} \tag{10}$$

neglecting ψ^4 and $(1-q^2/U^2)^3$; and this can be written

$$C_p = (-U^{-2}Q_0+\tfrac{1}{4}M^2U^{-4}Q_0^2)+\psi(-U^{-2}Q_1+\tfrac{1}{2}M^2U^{-4}Q_0\,Q_1)\cos\theta+$$
$$+\psi^2(-U^{-2}Q_2+\tfrac{1}{2}M^2U^{-4}Q_0\,Q_2+\tfrac{1}{4}M^2U^{-4}Q_1^2\cos^2\theta)+$$
$$+\psi^3(-U^{-2}Q_3+\tfrac{1}{2}M^2U^{-4}Q_0\,Q_3+\tfrac{1}{2}M^2U^{-4}Q_1\,Q_2\cos\theta). \qquad (11)$$

The force on the body along the axis is

$$D_1 = \tfrac{1}{2}\rho_0\,U^2\int\limits_0^{2\pi}d\theta\int\limits_0^{l}R(x)R'(x)C_p\,dx \qquad (12)$$

if the body extends from $x=0$ to $x=l$: to this a base-drag D_2 must be added when there is a flat base. The force perpendicular to the axis (which by symmetry is in the direction of the stream-component) is

$$L_1 = -\tfrac{1}{2}\rho_0\,U^2\int\limits_0^{2\pi}\cos\theta\,d\theta\int\limits_0^{l}R(x)C_p\,dx. \qquad (13)$$

The true lift (perpendicular to the stream) is then

$$L = L_1\cos\psi-(D_1+D_2)\sin\psi, \qquad (14)$$

and the true drag (parallel to the stream) is

$$D = (D_1+D_2)\cos\psi+L_1\sin\psi. \qquad (15)$$

The moment of the aerodynamic forces about an axis through the nose perpendicular both to the stream-direction and to the axis of the body is (in the sense tending to decrease the angle of yaw)

$$M = -\tfrac{1}{2}\rho_0\,U^2\left(\int\limits_0^{2\pi}\cos\theta\,d\theta\int\limits_0^{l}xR(x)C_p\,dx+\int\limits_0^{2\pi}\cos\theta\,d\theta\int\limits_0^{l}R^2(x)R'(x)C_p\,dx\right), \qquad (16)$$

the two terms being the moments due to forces perpendicular and parallel to the axis respectively. The expressions (12) to (16) are all exact.

3. First approximate solution for $\phi_0, \phi_1, \phi_2, \phi_3$

We now solve the equations near the body to a first approximation. If t is the maximum thickness of the body, we write '$O(t^m)$' henceforth to mean '$O(t^m\log^k t^{-1})$, for some k, when $r = O(t)$'.

Then equation (5) can be written $\nabla^2\phi_0 = O(t^2)$, to which a solution is easily verified to be $\phi_0 = K(x)+A(x)\log r+O(t^4)$, where K and A are $O(t^2)$. The boundary condition (5 a) becomes $AR^{-1} = UR'+O(t^3)$, so that $A = RR'$. To find K we observe that the *linearized* form of equation (5), that is, $\nabla^2\phi_0 = M^2\phi_{0xx}$, must have a solution beginning

$$\phi_0 = K+A\log r+O(r^2\log r^{-1}).$$

In Heaviside notation, with $p^{-1} = \int\limits_0^x dx$ and $\alpha = \sqrt{(M^2-1)}$, this linearized equation becomes

$$\frac{d^2\phi_0}{dr^2}+\frac{1}{r}\frac{d\phi_0}{dr}-\alpha^2p^2\phi_0 = 0, \qquad (17)$$

since $\phi_0 = \partial\phi_0/\partial x = 0$ at $x = 0$. By condition 'C' and the asymptotic forms of the Bessel functions K_0 and I_0, the solution of (17) that we require must have the form $K_0(\alpha pr)f(x)$; and for small r this is approximately $\{\log(2/\alpha pr)-\gamma\}f(x)$,† or (using the Heaviside representation of a Faltung integral and the fact that $\log x = -\log p-\gamma$)

$$\phi_0 = \int\limits_0^x \log\frac{2(x-y)}{\alpha r}\,df(y). \tag{18}$$

This is $K+A\log r$ if and only if $A = -f(x)$. Hence K is given in terms of A by the relation

$$K(x) = \int\limits_0^x \log\frac{\alpha}{2(x-y)}A'(y)\,dy. \tag{19}$$

Having obtained the relation $\phi_0 = K+A\log r+O(t^4)$, we see from (3) that

$$Q_0 = 2U(K'+A'\log r)+\frac{A^2}{r^2}+O(t^4). \tag{20}$$

Similarly equation (6) can be written $\nabla^2\phi_1-r^{-2}\phi_1 = O(t)$, to which a solution is $\phi_1 = Br^{-1}+Er+O(t^3)$. The boundary condition (6 a) is satisfied if $O(t^2)$ be neglected by taking $B = UR^2$, $E = O(t^2)$. E will be found in § 4 by a process similar to that by which K was found: at present we need merely observe that $\phi_1 = Br^{-1}+O(t^3)$ and so by (3),

$$Q_1 = 2U\frac{B'}{r}+2\left(U-\frac{B}{r^2}\right)\frac{A}{r}+O(t^3) = \frac{6UA}{r}-\frac{2AB}{r^3}+O(t^3), \tag{21}$$

since $2A = B'$. We can also write, if we assume that $\phi_2 = O(t^2)$,

$$\begin{aligned} Q_2 &= \left(-\frac{2UB}{r^2}+\frac{B^2}{r^4}\right)\cos^2\theta+\left(2U\frac{B}{r^2}+\frac{B^2}{r^4}\right)\sin^2\theta+O(t^2) \\ &= \frac{B^2}{r^4}-\frac{2UB}{r^2}\cos 2\theta+O(t^2). \end{aligned} \tag{22}$$

Equation (7) can now be written

$$\begin{aligned} 2a_0^2\nabla^2\phi_2 &= \frac{A}{r}\left(-\frac{4B^2}{r^5}+\frac{4UB}{r^3}\cos 2\theta\right)+\left(U-\frac{B}{r^2}\right)\left(-\frac{6UA}{r^2}+\frac{6AB}{r^4}\right)\cos^2\theta+ \\ &\quad +U\left(\frac{2BB'}{r^4}-\frac{2UB'}{r^2}\cos 2\theta\right)+r^{-2}\left(Ur+\frac{B}{r}\right)\left(\frac{6UA}{r}-\frac{2AB}{r^3}\right)\sin^2\theta+O(t^2) \\ &= -AB^2\frac{8+2\cos 2\theta}{r^6}+UAB\frac{12+8\cos 2\theta}{r^4}-10U^2A\frac{\cos 2\theta}{r^2}+O(t^2). \end{aligned} \tag{23}$$

† Here γ represents Euler's constant. This will not cause confusion since in the remainder of the paper the adiabatic index will not occur.

A solution is

$$2a_0^2\phi_2 = -AB^2\frac{\frac{1}{2}+\frac{1}{6}\cos 2\theta}{r^4}+UAB\frac{3-2\cos 2\theta\log r}{r^2}+$$
$$+\tfrac{5}{2}U^2A\cos 2\theta+Hr^{-2}\cos 2\theta+L\log r+N+O(t^4), \quad (24)$$

where H, L, N are arbitrary functions of x. The boundary condition $(7a)$ becomes

$$4AB^2\frac{\frac{1}{2}+\frac{1}{6}\cos 2\theta}{R^5}-2UAB\frac{3-2\cos 2\theta\log R+\cos 2\theta}{R^3}-\frac{2H\cos 2\theta}{R^3}+\frac{L}{R}$$
$$= -a_0^2\,UR'+O(t^3), \quad (25)$$

from which we deduce

$$H = \tfrac{1}{3}AB^2R^{-2}+UAB(2\log R-1)$$

and

$$L = -a_0^2\,URR'+6UABR^{-2}-2AB^2R^{-4}.$$

The value of N could be found in a similar manner to that in which K was found, but this is not required. We can rewrite (24) as

$$\phi_2 = -\tfrac{1}{2}UM^2R^5R'\,\frac{\frac{1}{2}+\frac{1}{6}\cos 2\theta}{r^4}+UM^2R^3R'\,\frac{\frac{3}{2}+\cos 2\theta(\log R-\log r-\frac{1}{3})}{r^2}+$$
$$+\tfrac{5}{4}UM^2RR'\cos 2\theta+\tfrac{1}{2}U(4M^2-1)RR'\log r+\tfrac{1}{2}a_0^{-2}N+O(t^4), \quad (26)$$

noticing that $\phi_2 = O(t^2)$, as was assumed above.

Finally, equation (8) can be written

$$2a_0^2\nabla^2\phi_3$$
$$= \left(U-\frac{B}{r^2}\right)\left(-\frac{4B^2}{r^5}+\frac{4UB}{r^3}\cos 2\theta\right)\cos\theta-r^{-2}\left(Ur+\frac{B}{r}\right)\left(\frac{4UB}{r^2}\sin 2\theta\right)\sin\theta+$$
$$+O(t)$$
$$= 4B^3\frac{\cos\theta}{r^7}-8UB^2\frac{\cos\theta}{r^5}+4U^2B\frac{\cos 3\theta}{r^3}+O(t). \quad (27)$$

A solution is

$$2a_0^2\phi_3 = \tfrac{1}{6}B^3\frac{\cos\theta}{r^5}-UB^2\frac{\cos\theta}{r^3}-\tfrac{1}{2}U^2B\frac{\cos 3\theta}{r}+P\frac{\cos 3\theta}{r^3}+Q\frac{\cos\theta}{r}+O(t^3). \quad (28)$$

Here P and Q are arbitrary. Inserting (28) in the boundary condition $(8a)$ we deduce that

$$-\frac{a_0^2}{3}U\cos\theta-\frac{5}{6}\frac{B^3}{R^6}\cos\theta+\frac{3UB^2}{R^4}\cos\theta+\tfrac{1}{2}U^2\frac{B}{R^2}\cos 3\theta-\frac{3P}{R^4}\cos 3\theta-\frac{Q}{R^2}\cos\theta$$
$$= O(t^2), \quad (29)$$

or

$$Q = \frac{13M^2-2}{6}a_0^2\,UR^2, \qquad P = \tfrac{1}{6}U^3R^4. \quad (30)$$

Hence

$$\phi_3 = \tfrac{1}{12}UM^2R^6\frac{\cos\theta}{r^5}+\tfrac{1}{12}UM^2R^4\frac{\cos 3\theta-6\cos\theta}{r^3}+$$
$$+\tfrac{1}{12}UR^2\frac{(13M^2-2)\cos\theta-3M^2\cos 3\theta}{r}, \quad (31)$$

neglecting $O(t^3)$.

Hence, if $O(t^3)$ be neglected, we have on the body $\{r = R(x)\}$, by (3),

$$\begin{aligned}Q_3 &= 2U[\tfrac{1}{2}UM^2R'\cos\theta+\tfrac{1}{3}UM^2R'(\cos 3\theta-6\cos\theta)+\\ &\quad+\tfrac{1}{6}UR'\{(13M^2-2)\cos\theta-3M^2\cos 3\theta\}]-2U^2R'\cos\theta+\\ &\quad+2UR'[-\tfrac{5}{12}UM^2\cos\theta-\tfrac{1}{4}UM^2(\cos 3\theta-6\cos\theta)-\\ &\quad-\tfrac{1}{12}U\{(13M^2-2)\cos\theta-3M^2\cos 3\theta\}-\tfrac{1}{6}U\cos\theta]-\\ &\quad-2.2UR\{\tfrac{1}{6}UM^2RR'\sin 2\theta-2UM^2RR'(-\tfrac{1}{3})\sin 2\theta-\\ &\quad-\tfrac{5}{2}UM^2RR'\sin 2\theta\}R^{-2}\sin\theta\\ &= U^2R'\{(\tfrac{14}{3}M^2-\tfrac{8}{3})\cos\theta-\tfrac{11}{3}M^2\cos 3\theta\}. \end{aligned} \quad (32)$$

We now apply equations (11), (20), (21), (22), and (32) to obtain that, at a point on the body,

$$\begin{aligned}C_p &= \{-2U^{-1}(K'+A'\log R)-R'^2\}+\psi\cos\theta(-4R')+\psi^2(-1+2\cos 2\theta)+\\ &\quad+\psi^3[R'\{(-\tfrac{14}{3}M^2+\tfrac{8}{3})\cos\theta+\tfrac{11}{3}M^2\cos 3\theta\}+\tfrac{1}{2}M^2\,4R'(1-2\cos 2\theta)\cos\theta],\end{aligned} \quad (33)$$

if $O(\psi^4)$, $O(\psi^3t^3)$, $O(\psi^2t^2)$, $O(\psi t^3)$, and $O(t^4)$ are neglected. The resulting value of D_1, by (12), is

$$D_0+\tfrac{1}{2}\rho_0 U^2 2\pi\int_0^l R(x)R'(x)(-\psi^2)\,dx, \quad (34)$$

where $D_0/\pi\rho_0 U^2$ is

$$\begin{aligned}&-\int_0^l RR'\left(2U^{-1}\frac{d}{dx}\int_0^x \log\frac{1}{x-y}A'(y)\,dy+2U^{-1}A'\log\frac{\alpha R}{2}+R'^2\right)dx\\ &= \frac{2}{U}R_0R_0'\int_0^l \log(l-y)A'(y)\,dy-\frac{2}{U^2}\int_0^l A'(x)\,dx\int_0^x \log(x-y)A'(y)\,dy-\\ &\qquad -\int_0^l \frac{d}{dx}\left(R^2R'^2\log\frac{\alpha R}{2}\right)dx\\ &= \frac{1}{U^2}\int_0^l\int_0^l A'(x)A'(y)\log\frac{1}{|x-y|}\,dxdy-\frac{2}{U}R_0R_0'\int_0^l A'(y)\log\frac{1}{l-y}\,dy+\\ &\qquad +R_0^2R_0'^2\log\frac{2}{\alpha R_0},\end{aligned} \quad (35)$$

where R_0, R_0' are the values of $R(l)$, $R'(l)$. Expression (35) is of course the usual one for the drag at zero yaw, reducing to the first term only when either R_0 or R_0' is zero. By (34) we see that the addition to D_1 due to yaw is $-\frac{1}{2}\rho_0 U^2\pi R_0^2\psi^2$ to a first approximation.

The value of L_1, by (13), is

$$\tfrac{1}{2}\rho_0 U^2\pi\left[\psi\int\limits_0^l R(x)\{4R'(x)\}\,dx+\psi^3\int\limits_0^l R(x)(\tfrac{14}{3}M^2-\tfrac{8}{3})R'(x)\,dx\right]$$

$$= \tfrac{1}{2}\rho_0 U^2\pi R_0^2\{2\psi+(\tfrac{7}{3}M^2-\tfrac{4}{3})\psi^3\}. \quad (36)$$

Here terms $O(\psi t^4)$ are neglected. The true lift L is (36) multiplied by $(1-\frac{1}{2}\psi^2)$ plus terms (from the drag) of orders which we are neglecting. The correction to the lift coefficient $C_L = 2\psi$ for thin bodies at higher angles of yaw therefore is $\frac{7}{3}\alpha^2\psi^3$. The percentage correction is $\frac{7}{6}\alpha^2\psi^2$; the most important thing to notice about this is that it is *small*—e.g. for $M = 2$, $\psi = 5°$, it is 2·7 per cent. Thus we see that for thin bodies of revolution the lift coefficient is closely a linear function of angle of incidence.

The true drag D, by (15), is $D_1+D_2+\frac{1}{2}\rho_0 U^2\pi R_0^2 2\psi^2$ to the orders we are contemplating; thus we can write $C_D = C_{D_0}+\psi^2$ for thin bodies with a flat base: for bodies pointed at both ends, however, C_D is not changed to order ψ^2.

The moment is easily seen by (16) to be

$$\tfrac{1}{2}\rho_0 U^2\pi\{4\psi+(\tfrac{14}{3}M^2-\tfrac{8}{3})\psi^3\}\int\limits_0^l xR(x)R'(x)\,dx, \quad (37)$$

if terms in ψt^4 and $\psi^3 t^4$ be neglected. The percentage change in C_M for thin bodies at higher angles of yaw is therefore $\dfrac{7M^2-4}{6}\psi^2$. This also is small; for $M = 2$, $\psi = 5°$, it is 3·05 per cent.

4. Second approximation to ϕ_1

Equation (6) can be written

$$2a_0^2(\nabla^2\phi_1-r^{-2}\phi_1) = \left(U-\frac{B}{r^2}\right)\left(2U\frac{A'}{r}-\frac{2A^2}{r^3}\right)+\frac{A}{r}\left(-\frac{6UA}{r^2}+\frac{6AB}{r^4}\right)+$$

$$+U\left(\frac{6UA'}{r}-2\frac{AB'+A'B}{r^3}\right)+O(t^3)$$

$$= \frac{8A^2B}{r^5}-\frac{12UA^2+4UA'B}{r^3}+\frac{8U^2A'}{r}+O(t^3). \quad (38)$$

This can be solved by assuming that

$$\phi_1 = \frac{B}{r} + \left(\frac{C}{r^3} + \frac{G}{r} + \frac{D\log r}{r} + Er + Fr\log r\right) + O(t^5), \tag{39}$$

if $\quad C = A^2B/2a_0^2 = \frac{1}{2}UM^2R^4R'^2, \qquad D = UM^2(4R^2R'^2 + R^3R'')$

and $F = (2M^2-1)A'$. We obtain G from the boundary condition (6 a) which takes the form

$$U - \frac{B}{R^2} - \frac{3C}{R^4} - \frac{G}{R^2} + D\frac{1-\log R}{R^2} + E + F(1+\log R) = R'\frac{B'}{R} + O(t^4), \tag{40}$$

giving

$$G = \frac{-3C}{R^2} + D(1-\log R) + ER^2 + FR^2(1+\log R) - B'RR'. \tag{41}$$

The determination of E is more difficult. If equation (6) be linearized, it takes the form

$$2a_0^2(\nabla^2\phi_1 - r^{-2}\phi_1) = 4U^2\phi_{0xr} + 2U^2\phi_{1xx},$$

or

$$\frac{\partial^2\phi_1}{\partial r^2} + \frac{1}{r}\frac{\partial\phi_1}{\partial r} - \alpha^2\frac{\partial^2\phi_1}{\partial x^2} - \frac{1}{r^2}\phi_1 = 2M^2\phi_{0xr}. \tag{42}$$

This is the equation satisfied by the coefficient of t^2 if ϕ_1 is expanded in ascending powers of t^2. The ϕ_0 on the right may be replaced by the linearized ϕ_0 which is $-K_0(\alpha pr)A(x)$. Now equation (42) must be satisfied by those terms in (39) whose coefficients are $O(t^2)$, i.e. $Br^{-1} + Er + Fr\log r$, together with terms involving higher powers of r. Thus to determine E we must find the necessary relation between the coefficients of r^{-1} and r in the expansion for small r of a solution of (42) which satisfies condition 'C' at infinity. To do this we write (42) in Heaviside form (remembering that $\phi_1 = \partial\phi_1/\partial x = A(x) = 0$ at $x = 0$):

$$\frac{\partial^2\phi_1}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} - \left(\alpha^2p^2 + \frac{1}{r^2}\right)\phi_1 = 2M^2\alpha p^2K_1(\alpha pr)A(x). \tag{43}$$

(Watson's notation $K_1(z) = -K_0'(z)$ is used.) A particular integral satisfying condition 'C' (i.e. behaving like $e^{-\alpha pr}$ not $e^{\alpha pr}$ at infinity) is

$$\phi_1 = -K_1(\alpha pr)\int_0^r \{2M^2\alpha p^2K_1(\alpha ps)A(x)\}sI_1(\alpha ps)\,ds - $$
$$-I_1(\alpha pr)\int_r^\infty \{2M^2\alpha p^2K_1(\alpha ps)A(x)\}sK_1(\alpha ps)\,ds, \tag{44}$$

since the Wronskian of I_1 and K_1 is $-1/r$. Equation (44) can be rewritten

$$\phi_1 = \left\{K_1(\alpha pr)\int_0^r sK_1(\alpha ps)I_1(\alpha ps)\,ds + I_1(\alpha pr)\int_r^\infty sK_1^2(\alpha ps)\,ds\right\}[-2\alpha p^2M^2A(x)]. \tag{45}$$

The expression in curly brackets is, for small r, equal to

$$\frac{1}{\alpha pr}\int\limits_0^r s\Big(\frac{1}{\alpha ps}\Big)\tfrac{1}{2}\alpha ps\,ds+\tfrac{1}{2}\alpha pr\Big[-\Big\{sK_1(\alpha ps)\frac{K_0(\alpha ps)}{\alpha p}\Big\}_r^\infty-\int\limits_r^\infty sK_0^2(\alpha ps)\,ds\Big]+ \\ +O(r^3\log r^{-1}). \quad (46)$$

But $\int\limits_0^\infty sK_0^2(\alpha ps)\,ds = 1/2\alpha^2p^2$, so (46) is

$$\frac{r}{4\alpha p}+\tfrac{1}{2}\alpha pr\Big\{\frac{1}{\alpha^2p^2}\Big(\log\frac{2}{\alpha pr}-\gamma\Big)-\frac{1}{2\alpha^2p^2}\Big\}+O(r^3\log^2 r^{-1}), \quad (47)$$

and (45) is

$$\phi_1 = -pr\Big(\log\frac{2}{\alpha pr}-\gamma\Big)M^2A(x)+O(r^3\log^2 r^{-1}). \quad (48)$$

To (48) can be added (in order to make the coefficient of r^{-1} correct) the expression $2\alpha K_1(\alpha pr)A(x)$, which is finite at infinity and for small r is asymptotic to $2A(x)/pr = B(x)/r$. The result is

$$\phi_1 = \Big[2\alpha\Big\{\frac{1}{\alpha pr}+\tfrac{1}{2}\alpha pr\Big(\log\frac{\alpha pr}{2}+\gamma-\tfrac{1}{2}\Big)\Big\}+M^2pr\Big(\log\frac{\alpha pr}{2}+\gamma\Big)\Big]A(x), \quad (49)$$

in which the coefficient of $1/r$ is B and that of r is

$$E = \Big\{(2M^2-1)p\Big(\log\frac{\alpha p}{2}+\gamma\Big)-\tfrac{1}{2}\alpha^2p\Big\}A(x)$$

$$= -\tfrac{1}{2}\alpha^2A'(x)+(2M^2-1)\frac{d}{dx}\int\limits_0^x A'(y)\log\frac{\alpha}{2(x-y)}\,dy. \quad (50)$$

We have now determined all the coefficients in (39) and can proceed to obtain the coefficient of $\psi\cos\theta$ in C_p. By (11), (3), and (6a), this, at a point on the body, is

$$-\frac{1}{U^2}\Big\{2(U+K'+A'\log R)\Big(\frac{B'}{R}+\frac{C'}{R^3}+\frac{G'}{R}+\frac{D'\log R}{R}+E'R+F'R\log R\Big)+ \\ +\frac{2R'B'}{R}\frac{A}{R}\Big\}+\tfrac{1}{2}M^2U^{-4}\Big(2U(K'+A'\log R)+\frac{A^2}{R^2}\Big)\Big(\frac{6UA}{R}-\frac{2AB}{R^3}\Big)+O(t^5)$$

$$= -\frac{2}{U}\Big(\frac{B'}{R}+\frac{C'}{R^3}+\frac{G'}{R}+\frac{D'\log R}{R}+E'R+F'R\log R\Big)+ \\ +\frac{4\alpha^2R'}{U}(K'+A'\log R)+2(M^2-2)R'^3+O(t^5). \quad (51)$$

The coefficient of ψ in $L_1/\pi\rho_0 U^2$ is therefore (by (13))

$$U^{-1}\left(\int_0^l B'\,dx+\int_0^l \frac{C'}{R^2}\,dx+\int_0^l G'\,dx+\int_0^l D'\log R\,dx+\int_0^l E'R^2\,dx+\right.$$

$$\left.+\int_0^l F'R^2\log R\,dx\right)-\frac{2\alpha^2}{U^2}\int_0^l A(K'+A'\log R)\,dx+(2-M^2)\int_0^l RR'^3\,dx$$

$$=R_0^2+\tfrac{1}{2}M^2R_0^2R_0'^2+\int_0^l M^2RR'^3\,dx-\tfrac{3}{2}M^2R_0^2R_0'^2+M^2(4R_0^2R_0'^2+R_0^3R_0'')+$$

$$+\frac{2R_0^2}{U}\left(-\tfrac{1}{2}\alpha^2A'(l)+(2M^2-1)\frac{d}{dl}\int_0^l A'(y)\log\frac{\alpha}{2(l-y)}\,dy\right)+$$

$$+R_0^2(1+2\log R_0)(2M^2-1)(R_0'^2+R_0R_0'')-2R_0^2R_0'^2-$$

$$-M^2\int_0^l(4RR'^3+R^2R'R'')\,dx-$$

$$-\frac{2}{U^2}\int_0^l A(x)\left(-\tfrac{1}{2}\alpha^2A'(x)+(2M^2-1)\frac{d}{dx}\int_0^x A'(y)\log\frac{\alpha}{2(x-y)}\,dy\right)dx-$$

$$-(2M^2-1)\int_0^l(R'^2+RR'')R(1+2\log R)R'\,dx-$$

$$-\frac{2\alpha^2}{U^2}\int_0^l A(x)\left(\frac{d}{dx}\int_0^x\log\frac{1}{x-y}A'(y)\,dy+A'(x)\log\frac{\alpha R}{2}\right)dx+$$

$$+(2-M^2)\int_0^l RR'^3\,dx$$

$$=R_0^2+\frac{3M^2-2}{U^2}\int_0^l\int_0^l A'(x)A'(y)\log\frac{\alpha R_0}{2|x-y|}\,dxdy-$$

$$-\frac{2(3M^2-2)R_0R_0'}{U}\int_0^l A'(y)\log\frac{\alpha R_0}{2(l-y)}\,dy+$$

$$+\frac{2(2M^2-1)}{U}R_0^2\frac{d}{dl}\int_0^l A'(y)\log\frac{\alpha R_0}{2(l-y)}\,dy+$$

$$+(3M^2-2)R_0^2R_0'^2+2M^2R_0^3R_0''. \tag{52}$$

The coefficient of ψ in the true lift (14) differs from that in L_1 by D_0+D_{20}, where D_0 is given by (35) and D_{20} is the base drag at zero yaw: thus it is†

$$\pi\rho_0 U^2\Big\{R_0^2+3\alpha^2\int\limits_0^l\int\limits_0^l \log\frac{\alpha R_0}{2|x-y|}\,d\{R(x)R'(x)\}\,d\{R(y)R'(y)\}-$$
$$-6\alpha^2R_0R_0'\int\limits_0^l \log\frac{\alpha R_0}{2(l-y)}\,d\{R(y)R'(y)\}+$$
$$+2(2M^2-1)R_0^2\frac{d}{dl}\left(\int\limits_0^l \log\frac{\alpha R_0}{2(l-y)}\,d\{R(y)R'(y)\}\right)+$$
$$+(3M^2-2)R_0^2R_0'^2+2M^2R_0^3R_0''\Big\}-D_{20}. \tag{53}$$

This gives the correction to the formula $L = \psi\pi\rho_0 U^2R_0^2$ due to terms of order ψt^4; earlier we found the correction due to terms of order ψ^3t^2 to be $\frac{7}{6}\alpha^2\psi^3\pi\rho_0 U^2R_0^2$.

For a body pointed at both ends we have simply

$$\frac{dL}{d\psi} = 3\alpha^2D_0, \tag{54}$$

by letting $R_0 \to 0$ in (53) and comparing with (35).

If the pseudo-lift L_1 on the portion $x < l'$ of the body is $L(l')$—given by $\psi\pi\rho_0 U^2$ times (52) with l' written for l, $R(l')$ for R_0 and $R'(l')$ for R_0'—then by (16) and (13) the moment M is

$$\int\limits_0^l \{x+R(x)R'(x)\}\frac{dL}{dx}\,dx = L_1(l+R_0R_0')-\int\limits_0^l L(x)(1+R'^2+RR'')\,dx$$
$$= L_1l+\psi\pi\rho_0U^2R_0^3R_0'-\int\limits_0^l L(x)\,dx-\psi\pi\rho_0U^2\int\limits_0^l R^2(R'^2+RR'')\,dx+$$
$$+O(\psi t^6)+O(\psi^3t^2), \tag{55}$$

an expression of which, for brevity, further simplification is omitted. The term in ψ^3t^2 was given in § 3. As an example we consider a cone of unit height and base diameter t. Then $l = 1$, $R(x) = \frac{1}{2}tx$. We find that

$$C_L = \frac{L}{\frac{1}{2}\rho_0U^2\frac{1}{4}\pi t^2}$$
$$= \left(2+\frac{M^2+1}{2}t^2\log\frac{\alpha t}{4}+\frac{3M^2-1}{4}t^2-C_{D_{20}}\right)\psi+\frac{7}{3}\alpha^2\psi^3+O(\psi t^4)+O(\psi^3t^2), \tag{56}$$

† Here and in (52) d/dl does not operate on R_0.

and

$$C_M = \frac{M}{\frac{1}{2}\rho_0 U^2 \frac{1}{4}\pi t^2 l}$$

$$= \tfrac{4}{3}\psi\left(1+\tfrac{1}{4}M^2t^2\log\frac{\alpha t}{4}+\tfrac{3}{8}M^2t^2\right)+\frac{14M^2-8}{9}\psi^3+O(\psi t^4)+O(\psi^3t^2). \quad (57)$$

Tables 1 and 2 give the values of $dC_L/d\psi$ at $\psi = 0$ (omitting the base drag term), and of $dC_M/d\psi$ at $\psi = 0$, respectively, for $M = 1{\cdot}5$, 2, 2·5, and 3 and for cones of semi-angle $\epsilon = 5°$, 10°, and 15°. The values for smaller ϵ may, of course, be expected to be more accurate.

C_L TABLE 1
(Cf. 2·000 at $\epsilon = 0°$.)

M	$\epsilon = 5°$	$\epsilon = 10°$	$\epsilon = 15°$
1·5	1·894	1·711	1·526
2·0	1·887	1·758	1·741
2·5	1·881	1·831	2·045
3·0	1·879	1·945	2·473

C_M TABLE 2
(Cf. 1·333 at $\epsilon = 0°$.)

$\epsilon = 5°$	$\epsilon = 10°$	$\epsilon = 15°$
1·298	1·257	1·247
1·289	1·270	1·348
1·282	1·308	1·524
1·279	1·375	1·790

It may be noted that our use of the exact equation does affect the terms in ψt^2 in C_L, since if the linearized equation be used (**1**), then $dC_{L_1}/d\psi$ is a function of αt only. The values of $dC_{L_1}/d\psi$ given by the present theory and by Tsien (**1**) are shown in Tables 3 and 4 respectively. (The good agreement at $M = 1{\cdot}5$ is seen to be merely due to the intersection near this point of widely diverging curves.)

TABLE 3

M	$\epsilon = 5°$	$\epsilon = 10°$	$\epsilon = 15°$
1·5	1·932	1·824	1·727
2·0	1·919	1·843	1·879
2·5	1·908	1·899	2·143
3·0	1·903	2·000	2·541

TABLE 4

$\epsilon = 5°$	$\epsilon = 10°$	$\epsilon = 15°$
1·944	1·832	1·700
1·887	1·692	1·492
1·828	1·565	1·324
1·768	1·451	1·186

5. Conclusion

The pressure coefficient on the body (by (33)) is

$$O(t^2)+O(\psi t)+O(\psi^2).$$

If, as seems likely, this is true even when the presence of the bow shock is taken into account, we may deduce that the strength of the shock is at most of this order.† Since the variation in entropy behind a shock is known to be of the order of the cube of its strength, we may conclude

† In a later paper it will be proved that when $\psi = 0$ the strength of the bow shock is actually $O(t^4)$.

that the error in using the adiabatic equations of motion behind the shock is at most

$$O(t^6)+O(\psi t^5)+O(\psi^2 t^4)+O(\psi^3 t^3)+O(\psi^4 t^2)+O(\psi^5 t)+O(\psi^6)$$

in the pressure coefficient. We have neglected terms of these orders and may therefore deduce that our expressions of order ψt^2 and ψ^3 in C_L and C_M additional to the known term of order ψ are correct. The results were given, respectively, at the end of § 4 and at the end of § 3.

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SUPERSONIC FLOW PAST SLENDER BODIES OF REVOLUTION THE SLOPE OF WHOSE MERIDIAN SECTION IS DISCONTINUOUS

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SUMMARY

The theory of supersonic flow around slender bodies of revolution, yawed or unyawed, with pointed or open bows, based on the linearized equation, is extended to the case when the meridian section of the outer surface has discontinuities in slope. Expressions for the pressure distribution on the surface are obtained. It is found that the drag coefficient is no longer independent of Mach number, and tends to zero more slowly than the square of the thickness of the body. The large pressure change behind a discontinuity is made up remarkably rapidly. The first approximation to the lift coefficient is unchanged.

1. Introduction

THE study of supersonic flow past bodies of revolution was initiated in 1931 by von Kármán (**1**), who used the 'linearized equation' of compressible flow. He advocated the exact solution of this equation under the exact boundary conditions—this involves the numerical solution of an integral equation—but also gave an approximate solution for the case of a body pointed at both ends. In 1945 the author (**2**) removed certain anomalies in the use of the linearized equation in this problem, and established von Kármán's approximate solution as a first approximation both for bodies pointed at both ends and for shells (in the region ahead of the flat base). Shapes with minimum drag were given and a theory was also evolved giving the flow outside any duct of shape approximating to a circular cylinder. The theory given for the flow inside the duct was incorrect. Broderick (**3**) (1947) has obtained a second approximation to the flow past pointed bodies and shells, using the exact equation of adiabatic flow: terms due to use of the more exact equation are shown to be important and it is inferred that, where the linearized equation is employed, not much is gained from proceeding beyond a first approximate solution. Meanwhile in 1938 Tsien (**4**) applied von Kármán's method to the yawed flow past a body of revolution. In a recent paper the author (**5**) obtained correct second approximations in this problem.

In all these approximate solutions for thin bodies it is necessary to assume that the slope of the meridian section be continuous and that its curvature be bounded. When the slope is discontinuous, the steps by

which the solution is reached are invalid, the solution is wrong, and, in fact, the expression which it gives for the drag of the body is infinite. Nor can these anomalies be removed by smoothing out the discontinuities by small curved portions whose curvature is allowed to increase indefinitely.

In this paper the theory of supersonic flow (both symmetrical and yawed) past thin bodies of revolution is modified to allow for discontinuities in slope. The theory is presented in such a general form that it also applies to flow outside slender ducts whose thickness is small compared with their length (but whose radius is not now necessarily approximately constant). Approximations to the pressure distribution, lift, drag, and moment are found for the flow past any slender body of revolution the outer boundary of whose meridian section is made up of a finite number of analytic arcs.

2. Lemmas obtained by Heaviside calculus

By ref. (**2**), §1, or ref. (**5**), §3, an approximate disturbance-velocity potential in the symmetrical flow will have the form

$$\phi_0 = \int\limits_{c_0}^{x-\alpha r} \frac{f(y)\,dy}{\sqrt{\{(x-y)^2-\alpha^2 r^2\}}} = K_0(\alpha p r) f(x), \tag{1}$$

where $\alpha = \sqrt{(M^2-1)}$, c_0 is any positive constant, p^{-1} is $\int\limits_0^x dx$ (in Heaviside notation) and K_0 is the Bessel function as defined by Watson. For small r, (1) is approximately equal to

$$\{\log(2/\alpha p r)-\gamma\} f(x) = \int\limits_0^x \log\left\{\frac{2(x-y)}{\alpha r}\right\} df(y) \tag{2}$$

(γ being Euler's constant), by the Heaviside representation of a Faltung integral and the fact that $\log x = -\log p - \gamma$. (We assume $f(x) = 0$ for $x < c_0$.) If $f(x)$ is continuous, and zero at $x = c_0$, the derivatives of (1) are

$$\frac{\partial \phi_0}{\partial x} = \int\limits_{c_0}^{x-\alpha r} \frac{f'(y)\,dy}{\sqrt{\{(x-y)^2-\alpha^2 r^2\}}} = pK_0(\alpha p r) f(x) \tag{3}$$

and $$\frac{\partial \phi_0}{\partial r} = -\frac{1}{r}\int\limits_{c_0}^{x-\alpha r} \frac{(x-y)f'(y)\,dy}{\sqrt{\{(x-y)^2-\alpha^2 r^2\}}} = -\alpha p K_1(\alpha p r) f(x) \sim -\frac{f(x)}{r} \tag{4}$$

as $r \to 0$. (On Watson's definition $K_1(z) = -K_0'(z) \sim z^{-1}$ as $z \to 0$.)

In ref. (**2**), §5, a function $U(x)$ is defined (and tabulated) such that

(equation (78)), if in (3) and (4) r is given the constant value R, then the relation

$$\frac{\partial\phi_0}{\partial x} = -\frac{1}{\alpha}\int_0^x U\left(\frac{x-y}{\alpha R}\right) d\left(\frac{\partial\phi_0(r,y)}{\partial r}\right) \tag{5}$$

subsists. Since, by (3) and (4), the relation must be

$$\frac{\partial\phi_0}{\partial x} = -\frac{K_0(\alpha pR)}{\alpha K_1(\alpha pR)}\frac{\partial\phi_0}{\partial r}, \tag{6}$$

we deduce that the Heaviside representation of $U(x)$ must be $K_0(p)/K_1(p)$. (This fact was first pointed out in an unpublished Admiralty Computing Service report. A complete theory of the flow outside and inside tubes of approximately cylindrical shape, based on this approach, is given by Ward, ref. (**6**).) Hence, for large x,

$$U_1(x) = \int_0^x U(y)\,dy = \frac{K_0(p)}{pK_1(p)} = \frac{\log(2/p)-\gamma}{p(p^{-1})} + O(p^2\log p^{-1})$$
$$= \log(2x)+O(x^{-2}\log x) \tag{7}$$

and

$$U(x) \sim \frac{1}{x}. \tag{8}$$

For the yawed flow at angle ψ to the axis, the disturbance-velocity potential can be written $\phi_0(r,x)+\psi\phi_1(r,x)\cos\theta+O(\psi^2)$ and an approximate equation for ϕ_1 is

$$\frac{\partial^2\phi_1}{\partial r^2}+\frac{1}{r}\frac{\partial\phi_1}{\partial r}-\frac{1}{r^2}\phi_1-\alpha^2\frac{\partial^2\phi_1}{\partial x^2} = 0, \tag{9}$$

a solution of which is

$$\phi_1 = \frac{1}{r}\int_{c_0}^{x-\alpha r} \frac{(x-y)g(y)\,dy}{\sqrt{\{(x-y)^2-\alpha^2r^2\}}} = \alpha K_1(\alpha pr)g(x). \tag{10}$$

The derivatives of ϕ_1, if $g(x)$ is continuous and zero at $x = c_0$, are

$$\frac{\partial\phi_1}{\partial x} = \frac{1}{r}\int_{c_0}^{x-\alpha r} \frac{(x-y)g'(y)\,dy}{\sqrt{\{(x-y)^2-\alpha^2r^2\}}} = \alpha pK_1(\alpha pr)g(x) \sim \frac{1}{r}g(x) \tag{11}$$

(as $r \to 0$) and

$$\frac{\partial\phi_1}{\partial r} = -\frac{1}{r^2}\int_{c_0}^{x-\alpha r} \frac{(x-y)^2g'(y)\,dy}{\sqrt{\{(x-y)^2-\alpha^2r^2\}}} = \alpha^2pK_1'(\alpha pr)g(x) \sim -\frac{1}{r^2}\int_{c_0}^{x} g(y)\,dy. \tag{12}$$

We shall need to solve the integral equation

$$x-c-\alpha R = \int_c^{x-\alpha R} \frac{(x-y)^2g'(y)\,dy}{\sqrt{\{(x-y)^2-\alpha^2R^2\}}} = -\alpha^2pR^2K_1'(\alpha pR)g(x) \tag{13}$$

(for $x > c+\alpha R$), and find $\partial\phi_1/\partial x$, when $r = R$, for the $g(x)$ which is the solution. Clearly we have

$$\frac{\partial\phi_1}{\partial x} = -\alpha p K_1(\alpha pR)\frac{p^{-1}e^{-c-\alpha R}}{\alpha^2 pR^2 K_1'(\alpha pR)} = \frac{1}{R}F\left(\frac{x-c-\alpha R}{\alpha R}\right), \tag{14}$$

where
$$F(x) = -K_1(p)/pK_1'(p) = 1+O(p^2\log p^{-1}) \tag{15}$$

as $p \to 0$. Thus

$$F(x) \to 1 \text{ as } x \to \infty, \quad \text{and} \quad \int\limits_0^\infty \{1-F(x)\}\,dx = 0. \tag{16}$$

The function $F(x)$ was first used by Ward, who denotes it by $L(x)$ (see ref. **6**): a table is given in § 4.

3. The symmetrical flow

Using cylindrical polar coordinates (r,θ,x), let $r = R(x)$ be the equation of the outer boundary of the meridian section of the body of revolution considered; let it extend from $x = a_0$ to $x = a_n$ and have maximum thickness t. Let $R(x)$ be continuous in $a_0 \leqslant x \leqslant a_n$ and analytic in each interval $a_0 < x < a_1$, $a_1 < x < a_2$, ..., $a_{n-1} < x < a_n$. Put†

$$R'(a_i+0)-R'(a_i-0) = b_i$$

(interpreting this for $i = 0$ and $i = n$ by the convention that $R'(x) = 0$ for $x < a_0$ and $x > a_n$); put also $R(a_i) = R_i$, $a_i-\alpha R_i = c_i$ (where $\alpha = \sqrt{(M^2-1)}$, M being the Mach number of the undisturbed stream), $S(x) = \pi R^2(x)$ (this is the cross-sectional area at distance x from the origin); then we must have $S'(a_i+0)-S'(a_i-0) = 2\pi b_i R_i$.

When there is no discontinuity the velocity potential is given (to a first approximation) by equation (1) with $f(x) = -S'(x)/2\pi$. We take as our corrected value of $f(x)$ the expression

$$f(x) = -\frac{1}{2\pi}\int\limits_{a_0}^{x} S''(y)\,dy - \sum_{i=0}^{n} f_i(x), \tag{17}$$

where $f_i(x)$ is zero for $x \leqslant c_i$ and continuous for $x \geqslant c_i$. The integral here (which can be written $p^{-1}S''(x)$) is *not* a Stieltjes integral with $S''(y)\,dy$ written for $dS'(y)$—we shall always write Stieltjes integrals as such throughout this paper—but is an ordinary Lebesgue integral which ignores the points $y = a_i$ where the integrand is not defined. Thus in the interval $a_j < x < a_{j+1}$ it is $S'(x)-\sum\limits_{i=0}^{j} 2\pi b_i R_i$.

The boundary condition to our order of approximation is that $\partial\phi_0/\partial r = R'(x)$ when $r = R(x)$. Using the approximation (4) for the part

† Dashes denote differentiation with respect to x.

of $\partial\phi_0/\partial r$ due to the first term in $f(x)$ we can write the condition (in the interval $a_j < x < a_{j+1}$) as

$$R'(x) = \frac{1}{2\pi R(x)}\Big(2\pi R(x)R'(x) - \sum_0^j 2\pi b_i R_i\Big) + \\ + \frac{1}{R(x)}\sum_0^j \int\limits_{c_i}^{x-\alpha R(x)} \frac{(x-y)f_i'(y)\,dy}{\sqrt{\{(x-y)^2-\alpha^2R^2(x)\}}}. \tag{18}$$

Equation (18) can be satisfied for all j if, for all i, $f_i(x)$ satisfies

$$b_i R_i = \int\limits_{c_i}^{x-\alpha R(x)} \frac{(x-y)f_i'(y)\,dy}{\sqrt{\{(x-y)^2-\alpha^2R^2(x)\}}}. \tag{19}$$

The term P_i due to $f_i(x)$ in $\partial\phi_0/\partial x$ is then given by (3) with $-f_i$ written for f and $R(x)$ written for r. When x is *near* a_i, so that $R(x) \simeq R_i$, an approximate solution, by (5), is

$$P_i = -\frac{b_i}{\alpha}U\Big(\frac{x-a_i}{\alpha R_i}\Big) \tag{20}$$

(making the convention that $U(x) = 0$ for $x < 0$). When x is not near a_i equation (19) becomes, by (4), approximately $f_i(x) = b_i R_i$; and in

$$P_i = -\int\limits_{c_i}^{x-\alpha R(x)} \frac{f_i'(y)\,dy}{\sqrt{\{(x-y)^2-\alpha^2R^2(x)\}}} \tag{21}$$

the greater part of the integral comes from a small region near $y = c_1$ in which $f_i(y)$ rises rapidly from 0 to $b_i R_i$; so that we can write

$$P_i \simeq -\frac{b_i R_i}{x-a_i}. \tag{22}$$

Since (20) also satisfies this approximate equality (by (8)) when x is not near a_i, (20) can be taken as a good approximation to P_i for all x.

We deduce that the pressure distribution is given by

$$C_p = \frac{p-p_0}{\frac{1}{2}\rho_0 V^2} = -2\frac{\partial\phi_0}{\partial x} - \Big(\frac{\partial\phi_0}{\partial r}\Big)^2$$
$$= \frac{1}{\pi}\int\limits_{a_0}^{x-\alpha R(x)} \frac{S''(y)\,dy}{\sqrt{\{(x-y)^2-\alpha^2R^2(x)\}}} + \sum_{i=0}^{n}\frac{2b_i}{\alpha}U\Big(\frac{x-a_i}{\alpha R_i}\Big) - R'^2(x) \tag{23}$$

on the body, approximating as in ref. (**2**).

It should be noticed that (to our degree of approximation) as soon as each summed term in (23) attains its asymptotic form $2b_i R_i/(x-a_i)$, it becomes the term due to the discontinuity at $y = a_i$ in the Stieltjes integral

$$\frac{1}{\pi}\int\limits_{0}^{x-\alpha R(x)} \frac{dS'(y)}{\sqrt{\{(x-y)^2-\alpha^2R^2(x)\}}}, \tag{24}$$

which is the natural extension of the form which applies when discontinuities are absent. Thus we may say that each discontinuity (including the front if $S'(x) \neq 0$ there) affects appreciably only the region immediately behind it: there is a sudden change of pressure (a drop, in the case of an 'expansive' discontinuity); but in a distance of the order of the thickness t of the body the pressure is made up to what it would have been had the shape been continuous. That this effect can be appreciable is indicated by the fact that the sudden change in C_p of $2b_i/\alpha$ is of order t, while the other terms in (23) are $O(t^2 \log t^{-1})$ at most.

Writing D for the drag divided by $\frac{1}{2}\rho_0 V^2$, we have

$$D = \int\limits_0^\infty C_p\, S'(x)\, dx, \tag{25}$$

where C_p is given by (23). With $U_1(x)$ defined as in (7) the term

$$\int\limits_0^\infty U\left(\frac{x-a_i}{\alpha R_i}\right) S'(x)\, dx \tag{26}$$

becomes

$$-\alpha R_i \int\limits_{a_i}^\infty U_1\left(\frac{x-a_i}{\alpha R_i}\right) dS'(x). \tag{27}$$

Now by equation (7) we can write $|U_1(x) - \log(2x)| < A_0 x^{-2} \log x$ for $x > A_1$, while for $x < A_1$ we have $|U_1(x)| < A_2$, say. Hence (27) can be written

$$-\alpha R_i \int\limits_{a_i}^\infty \log\left(2\frac{x-a_i}{\alpha R_i}\right) dS'(x) + O\left[\int\limits_{a_i+A_1\alpha R_i}^\infty \frac{\alpha^3 R_i^3}{(x-a_i)^2} \log\left(\frac{x-a_i}{\alpha R_i}\right) |dS'(x)| + \right.$$
$$\left. + \int\limits_{a_i}^{a_i+A_1\alpha R_i} \alpha R_i \left(1 + \left|\log 2\frac{x-\alpha_i}{\alpha R_i}\right|\right) |dS'(x)|\right], \tag{28}$$

and the O-term is $O(t^4 \log t^{-1})$.

Again we have, by (2),

$$\int\limits_{a_0}^{x-\alpha R(x)} \frac{S''(y)\, dy}{\sqrt{\{(x-y)^2 - \alpha^2 R^2(x)\}}} = \int\limits_0^x \log \frac{2(x-y)}{\alpha R(x)}\, dS''(y), \tag{29}$$

correct to order t^2 except in small regions to the right of each discontinuity of length $O(t)$, where the discrepancy will not affect the integral (25) to order t^4. Multiplying (29) by $S'(x)$ and integrating from 0 to ∞, we obtain

$$\int\limits_0^\infty S'(x) S''(x) \log \frac{2}{\alpha R(x)}\, dx + \int\limits_0^\infty S'(x)\, dx \int\limits_0^x \log(x-y)\, dS''(y), \tag{30}$$

and the second term can be rewritten

$$\int_0^\infty dS''(y)\int_y^\infty S'(x)\log(x-y)\,dx = -\int_0^\infty S''(y)\,d\left(\int_y^\infty S'(x)\log(x-y)\,dx\right)$$

$$= -\int_0^\infty S''(y)\,dy\int_y^\infty \log(x-y)\,dS'(x). \tag{31}$$

Collecting terms we deduce that

$$D = \frac{1}{\pi}\int_0^\infty S'(x)S''(x)\log\frac{2}{\alpha R(x)}\,dx - \frac{1}{\pi}\int_0^\infty S''(y)\,dy\int_y^\infty \log(x-y)\,dS'(x) -$$

$$-\sum_{i=0}^n 2b_i R_i\int_{a_i}^\infty \log\left(2\frac{x-a_i}{\alpha R_i}\right)dS'(x) - \int_0^\infty S'(x)R'^2(x)\,dx. \tag{32}$$

The first and fourth terms combine (since $S'(x) = 2\pi R(x)R'(x)$) to give

$$\frac{1}{2\pi}\int_0^\infty \frac{d}{dx}\left\{S'^2(x)\log\frac{2}{\alpha R(x)}\right\}dx = \frac{1}{2\pi}\sum_{i=0}^n\left(\log\frac{2}{\alpha R_i}\right)\{S'^2(a_i-0)-S'^2(a_i+0)\}, \tag{33}$$

and this combines with the part of the third term

$$-\sum_{i=0}^n 2b_i R_i\int_{a_i}\log\frac{2}{\alpha R_i}\,dS'(x) = \frac{1}{\pi}\sum_{i=0}^n\left(\log\frac{2}{\alpha R_i}\right)\{S'(a_i+0)-S'(a_i-0)\}S'(a_i+0) \tag{34}$$

to give

$$\frac{1}{2\pi}\sum_{i=0}^n\left(\log\frac{2}{\alpha R_i}\right)\{S'(a_i+0)-S'(a_i-0)\}^2 = 2\pi\sum_{i=0}^n b_i^2 R_i^2\log\frac{2}{\alpha R_i}. \tag{35}$$

The remainder of (32) can be written

$$-\frac{1}{\pi}\int_0^\infty S''(y)\,dy\int_y^\infty S''(x)\log(x-y)\,dx - \sum_{i=0}^n 2b_i R_i\int_0^{a_i} S''(y)\log(a_i-y)\,dy -$$

$$-\sum_{i=0}^n 2b_i R_i\left\{\int_{a_i}^\infty S''(x)\log(x-a_i)\,dx + \sum_{j>i} 2\pi b_j R_j\log(a_j-a_i)\right\}$$

$$= \frac{1}{2\pi}\int_{a_0}^{a_n}\int_{a_0}^{a_n} S''(x)S''(y)\log\frac{1}{|x-y|}\,dxdy + \sum_{i=0}^n 2b_i R_i\int_{a_0}^{a_n} S''(x)\log\frac{1}{|x-a_i|}\,dx +$$

$$+\sum\sum_{j>i} 4\pi b_i b_j R_i R_j\log\frac{1}{|a_i-a_j|}, \tag{36}$$

and D is given by (35) plus (36).

The form of (36) is not surprising, being a natural extension of the form of D when discontinuities are absent. For consider the formal Stieltjes integral

$$\frac{1}{2\pi}\int\limits_0^\infty dS'(x)\int\limits_0^\infty \log\frac{1}{|x-y|}\,dS'(y)$$

$$=\frac{1}{2\pi}\int\limits_0^\infty dS'(x)\int\limits_0^\infty S''(y)\log\frac{1}{|x-y|}\,dy+\sum_{i=0}^{n}\frac{1}{2\pi}\int\limits_0^\infty 2\pi b_i\,R_i\log\frac{1}{|x-a_i|}\,dS'(x)$$

$$=\frac{1}{2\pi}\int\limits_{a_0}^{a_n} S''(x)\,dx\int\limits_{a_0}^{a_n} S''(y)\log\frac{1}{|x-y|}\,dy+2\sum_{i=0}^{n}b_i\,R_i\int\limits_{a_0}^{a_n}S''(x)\log\frac{1}{|x-a_i|}\,dx+$$

$$+\sum_{i=0}^{n}b_i\,R_i\sum_{j=0}^{n}2\pi b_j\,R_j\log\frac{1}{|a_j-a_i|}. \tag{37}$$

If the infinite terms with $i=j$ in the latter double sum are all discarded, (37) becomes identical with (36), which can therefore be appropriately termed the 'finite part' of the double Stieltjes integral. Using an asterisk to denote finite part we may then write

$$D=\frac{1}{2\pi}{}^*\int\limits_0^\infty\int\limits_0^\infty \log\frac{1}{|x-y|}\,dS'(x)\,dS'(y)+2\pi\sum_{i=0}^{n}b_i^2\,R_i^2\log\frac{2}{\alpha R_i}, \tag{38}$$

where the first term is defined by expression (36) and has properties similar to those of D when discontinuities are absent. The second term, however, is quite new and makes two important changes:

(i) D is now dependent on Mach number, decreasing as the latter increases;

(ii) D now contains terms in $t^4\log t^{-1}$ as well as t^4; the former are of course greater for small t.

As an example consider the flow past a double cone of thickness t, defined by the equations

$$R(x)=\begin{Bmatrix} tx & (0<x<\tfrac{1}{2}) \\ t(1-x) & (\tfrac{1}{2}<x<1)\end{Bmatrix}. \tag{39}$$

There is one discontinuity, $a_1=\frac{1}{2}$, where $b_1=-2t$, $R_1=\frac{1}{2}t$. We have $S''(x)=2\pi t^2$ for $0<x<1$. Equation (23) becomes

$$\frac{C_p}{t^2}=\begin{Bmatrix} 2\cosh^{-1}(1/\alpha t)-1 & (0<x<\tfrac{1}{2}) \\ 2\cosh^{-1}\left(\dfrac{x}{\alpha t(1-x)}\right)-1-\dfrac{4}{\alpha t}U\left(\dfrac{2x-1}{\alpha t}\right) & (\tfrac{1}{2}<x<1)\end{Bmatrix} \tag{40}$$

and equation (38) gives

$$C_D=\frac{D}{\frac{1}{4}\pi t^2}=8t^2\left(\log\frac{1}{\alpha t}-\frac{1}{2}\right). \tag{41}$$

C_D/t^2, and values of C_p/t^2 as a function of x, are given for five values of αt in Table I. The relative drop in pressure at $x = \frac{1}{2}$ is seen to be very

TABLE I

αt	$0 < x < 0.5$	0·50	0·55	0·60	0·65	0·70	0·75	0·80	0·85	0·90	0·95	$C_D t^{-2}$
0·05	+6·2	−73·6	−29·0	−13·5	−6·4	−2·4	+0·3	+2·3	+4·0	+5·6	+7·7	32·84
0·10	+5·0	−35·0	−20·2	−12·1	−7·0	−3·7	−1·2	+0·7	+2·5	+4·2	+6·2	14·42
0·15	+4·2	−22·5	−15·0	−10·0	−6·5	−3·8	−1·7	+0·0	+1·7	+3·3	+5·4	11·18
0·20	+3·6	−16·4	−11·8	− 8·4	−5·8	−3·6	−1·8	−0·3	+1·2	+2·8	+4·8	8·88
0·25	+3·1	−12·9	− 9·7	− 7·2	−5·1	−3·3	−1·8	−0·4	+1·0	+2·5	+4·5	7·09

great especially for small αt: the recovery is also very rapid. The maximum of dC_p/dx (which occurs just behind the discontinuity, after which it falls rapidly) is approximately $4/\alpha^2$, since $U'(0) = -\frac{1}{2}$. When M is near unity this may cause boundary layer separation or at least transition to turbulence: experiments with double-wedge aerofoils have not indicated separation as likely when the angle turned through expansively is small. In ref. (**2**), § 3, it is shown that for a parabolic shape the quantity C_D/t^2 is 10·67, for two other smooth shapes 11·64 and 12·5, and that the minimum possible value for a smooth body symmetrical about a central plane is 9·87. Thus we see that for a very thin body the discontinuity increases the drag, but that for thicker bodies this is not so. In fact we would be tempted to infer that for $\alpha t = 0{\cdot}20$ and $0{\cdot}25$ the discontinuity lessens the drag, were it not probable that terms of higher order than t^2 in C_D become important for these values.

We observe, in passing, the import of our results in the theory of cylindrical sound-waves produced by the expansion and contraction of a small cylinder: the mathematics is of course identical in the two theories. When the circumferential velocity is changed discontinuously by an amount V, the surface pressure changes by the relatively large amount ρVa; but this change is largely made up in an interval of time $O(R/a)$, where R is the radius of the cylinder: this interval is very short. Again, our expression for the drag of a given body of revolution gives the work done in expanding and contracting the small cylinder in a given manner.

4. The yawed flow

We first treat the yawed flow on the basis of equation (9), considering afterwards how far the use of a more exact equation will alter our results. When discontinuities are absent we have equation (10) with $g(x) = S'(x)/\pi$. We take as our corrected value of $g(x)$ the expression

$$g(x) = \frac{1}{\pi}\int\limits_{a_0}^{x} S''(y)\,dy + \sum_{i=0}^{n} g_i(x), \tag{42}$$

where each $g_i(x)$ is zero for $x \leqslant c_i$ and continuous for $x \geqslant c_i$. Our boundary condition to the first order (**5**) is that $\partial\phi_1/\partial r = -1$ when $r = R(x)$. Using the approximation (11) for the part of $\partial\phi_1/\partial r$ due to the first term in $g(x)$ we can write the condition (in the interval $a_j < x < a_{j+i}$) as

$$-1 = -\frac{1}{\pi R^2(x)}\Big\{\pi R^2(x)-\pi R_0^2-\sum_0^j 2\pi b_i R_i\,(x-a_i)\Big\}-$$

$$-\frac{1}{R^2(x)}\sum_0^j \int\limits_{c_i}^{x-\alpha R(x)} \frac{(x-y)^2 g_i'(y)\,dy}{\sqrt{\{(x-y)^2-\alpha^2R^2(x)\}}}, \qquad (43)$$

which can be satisfied for all j if, for all i, when $x > a_i$,

$$2b_i R_i(x-a_i)+[R_0^2] = \int\limits_{c_i}^{x-\alpha R(x)} \frac{(x-y)^2 g_i'(y)\,dy}{\sqrt{\{(x-y)^2-\alpha^2R^2(x)\}}}, \qquad (44)$$

the square brackets indicating that the term within them occurs only for $i = 0$. Hence by (14), when x is *near* a_i, so that we may replace $R(x)$ by R_i approximately, the term q_i in $r\partial\phi_1/\partial x$ (on the body) which contains g_i is

$$q_i = 2b_i\,R_i\,F\Big(\frac{x-a_i}{\alpha R_i}\Big)+\Big[\frac{R_0}{\alpha}F'\Big(\frac{x-a_0}{\alpha R_0}\Big)\Big]. \qquad (45)$$

When x is not near a_i, however, equation (44) becomes, by (12), approximately

$$2b_i\,R_i(x-a_i)+[R_0^2] = \int\limits_{c_i}^{x} g_i(y)\,dy, \qquad g_i(x) = 2b_i\,R_i, \qquad (46)$$

and we have

$$q_i = \int\limits_{c_i}^{x-\alpha R(x)} \frac{(x-y)g_i'(y)\,dy}{\sqrt{\{(x-y)^2-\alpha^2R^2(x)\}}} \simeq g_i(x) \simeq 2b_i\,R_i. \qquad (47)$$

Since (45) also satisfies the approximate equality (47) when x is not near a_i, by (16), we can take (45) as a good approximation to q_i for all x. We then have (see ref. **5**), to our degree of approximation, on the body,

$$\frac{dC_p}{d\psi} = -2\frac{\partial\phi_1}{\partial x}\cos\theta$$

$$= -2\frac{\cos\theta}{R(x)}\Big\{\frac{1}{\pi}\int\limits_{a_0}^{x} S''(y)\,dy+\sum_{i=0}^{n} 2b_i\,R_i\,F\Big(\frac{x-a_i}{\alpha R_i}\Big)+\frac{R_0}{\alpha}F'\Big(\frac{x-a_0}{\alpha R_0}\Big)\Big\}. \qquad (48)$$

The functions $F(x)$, $F'(x)$ are given in Table II, condensed from a fuller table in ref. (**6**). Since $F(0) = 0$, $dC_p/d\psi$ is continuous everywhere: as each F attains its asymptotic value 1 (which it does rapidly, though overshooting it at first) the first two terms in curly brackets in (48) become

simply $S'(x)/\pi$. The third term, which is only of importance near the front, where it predominates over the other two, is that occurring (by itself) in the theory of yawed flow past quasi-cylinders, due to Ward. (It is of course identically zero when $R_0 = 0$.)

Writing L for the lift divided by $\frac{1}{2}\rho_0 V^2$ we have (ref. 5), neglecting terms $O(t^4 \log t^{-1})$,

$$\frac{dL}{d\psi} = -\int\limits_0^{2\pi} \cos\theta\, d\theta \int\limits_{a_0}^{a_n} R(x)\frac{dC_p}{d\psi}\,dx$$

$$= 2\pi \int\limits_{a_0}^{a_n} \left\{\frac{1}{\pi}\int\limits_{a_0}^{x} S''(y)\,dy + \sum_{i=0}^{n} 2b_i R_i F\left(\frac{x-a_i}{\alpha R_i}\right) + \frac{R_0}{\alpha} F'\left(\frac{x-a_0}{\alpha R_0}\right)\right\} dx$$

$$= 2\int\limits_{a_0}^{a_n} S'(x)\,dx + 4\pi \sum_{i=0}^{n} b_i R_i \int\limits_{a_i}^{a_n} \left\{F\left(\frac{x-a_i}{\alpha R_i}\right) - 1\right\} dx + \frac{2\pi R_0}{\alpha}\int\limits_{a_0}^{a_n} F'\left(\frac{x-a_0}{\alpha R_0}\right) dx. \tag{49}$$

TABLE II

x	$F(x)$	$F'(x)$
0·0	0·000	1·000
0·4	0·359	0·796
0·8	0·638	0·600
1·2	0·843	0·428
1·6	0·985	0·287
2·0	1·076	0·176
2·4	1·129	0·093
2·8	1·154	0·034
3·2	1·159	−0·005
3·6	1·152	−0·029
4·0	1·138	−0·042
4·4	1·120	−0·046
4·8	1·101	−0·046
5·2	1·084	−0·042
5·6	1·068	−0·037
6·0	1·054	−0·031
6·4	1·043	−0·026
6·8	1·034	−0·020
7·2	1·027	−0·016
7·6	1·021	−0·012
8·0	1·017	−0·009

By use of (15)—the precise form of (16)—the integral from a_i to a_n is seen, by putting

$$x = a_i + \alpha R_i y,$$

to be $O(t^4 \log t^{-1})$. The last integral in (49) is

$$\alpha R_0 \int\limits_0^{\frac{a_n-a_0}{\alpha R_0}} F'(y)\,dy = \alpha R_0 F\left(\frac{a_n-a_0}{\alpha R_0}\right)$$

$$= \alpha R_0 + O(t^3 \log t^{-1}). \tag{50}$$

Hence neglecting terms $O(t^4 \log t^{-1})$, (49) becomes

$$2\{S(a_n) - S(a_0)\} + 2\pi R_0^2 = 2S(a_n). \tag{51}$$

This is the ordinary result for a thin body, which is seen to be true independently of discontinuities. (The other novelty is that we have established it for an open-ended body not necessarily of 'quasi-cylindrical' shape.) Though the discontinuities produce changes of pressure-distribution of the order we are considering, these integrate out to zero in the expression for the lift (as a result of equation (16)).

Similarly the moment is unchanged. In fact, writing M for the moment

about an axis in the plane of the nose (perpendicular to the stream) in the sense tending to decrease the angle of yaw, divided by $\frac{1}{2}\rho_0 V^2$, we have (see ref. **5**), neglecting terms $O(t^4 \log t^{-1})$,

$$\begin{aligned}\frac{dM}{d\psi} &= -\int_0^{2\pi} \cos\theta\, d\theta \int_{a_0}^{a_n} (x-a_0)R(x)\frac{dC_p}{d\psi}\,dx \\ &= 2\pi \int_{a_0}^{a_n} (x-a_0)\left\{\frac{1}{\pi}\int_{a_0}^{x} S''(y)\,dy + \sum_{i=0}^{n} 2b_i R_i F\left(\frac{x-a_i}{\alpha R_i}\right) + \frac{R_0}{\alpha} F'\left(\frac{x-a_0}{\alpha R_0}\right)\right\}dx \\ &= 2\int_{a_0}^{a_n} (x-a_0)S'(x)\,dx + 4\pi \sum_{i=0}^{n} b_i R_i \int_{a_i}^{a_n} (x-a_0)\left\{F\left(\frac{x-a_i}{\alpha R_i}\right)-1\right\}dx + \\ &\qquad + \frac{2\pi R_0}{\alpha}\int_{a_0}^{a_n} (x-a_0)F'\left(\frac{x-a_0}{\alpha R_0}\right)dx. \end{aligned} \tag{52}$$

The integral from a_i to a_n in (52) can be written

$$\int_0^{\frac{a_n-a_i}{\alpha R_i}} (a_i - a_0 + \alpha R_i y)\{F(y)-1\}\alpha R_i\,dy = O(t^2 \log^2 t^{-1}); \tag{53}$$

and the last integral in (52) is

$$\begin{aligned}\alpha^2 R_0^2 \int_0^{\frac{a_n-a_0}{\alpha R_0}} yF'(y)\,dy &= \alpha^2 R_0^2 \left\{\Big[y\{F(y)-1\}\Big]_0^{\frac{a_n-a_0}{\alpha R_0}} - \int_0^{\frac{a_n-a_0}{\alpha R_0}} \{F(y)-1\}\,dy\right\} \\ &= O(t^3 \log t^{-1}); \end{aligned} \tag{54}$$

hence finally (neglecting terms $O(t^4 \log^2 t^{-1})$)

$$\frac{dM}{d\psi} = 2\int_{a_0}^{a_n} (x-a_0)S'(x)\,dx = 2\int_{a_0}^{a_n} \{S(a_n)-S(x)\}\,dx. \tag{55}$$

Again the interest is that we have established (55) generally for thin bodies even with an open front. The aerodynamic centre is at a distance

$$\int_{a_0}^{a_n} \left(1 - \frac{S(x)}{S(a_n)}\right)dx \tag{56}$$

behind the front (by (51) and (55)).

We must point out here that, as was shown in ref. (**5**) (equation (42)), our equation (9) is not the true linearized equation for ϕ_1, which should contain a term $2M^2\partial^2\phi_0/\partial x\partial r$ on the right-hand side. However, using the value of ϕ_0 obtained in § 3, and solving the equation as in ref. (**5**), § 4, it

may be shown that additional terms in $dC_p/d\psi$ are all of order $t^3 \log t^{-1}$ which we are already neglecting. Similarly no change, except to this order, results from using a still more exact equation for ϕ_1 as in equation (38) of (5).

As an example, we consider the yawed flow past a double cone frustum of thickness t, defined by the equations

$$R(x) = \left\{\begin{matrix} \frac{1}{2}t(x+\frac{1}{2}) & (0 < x < \frac{1}{2}) \\ \frac{1}{2}t(\frac{3}{2}-x) & (\frac{1}{2} < x < 1) \end{matrix}\right\}. \tag{57}$$

Here $a_0 = 0$, with $b_0 = \frac{1}{2}t$, $R_0 = \frac{1}{4}t$, and $a_1 = \frac{1}{2}$, with $b_1 = -t$, $R_1 = \frac{1}{2}t$. Equation (48) becomes

$$-\alpha\frac{dC_p}{d\psi}\sec\theta = \left\{\begin{matrix} \{2\alpha tx+\alpha tF(4x/\alpha t)+F'(4x/\alpha t)\}/(x+\frac{1}{2}) \\ [2\alpha tx+\alpha tF(4x/\alpha t)+F'(4x/\alpha t)-4\alpha tF\{(2x-1)/\alpha t\}]/(\frac{3}{2}-x) \end{matrix}\right\} \tag{58}$$

and this quantity is tabulated below for two values of αt. The lift for this body satisfies $dL/d\psi = \pi t^2/8$, and the aerodynamic centre is at a

αt	$x = 0$	0·05	0·1	0·2	0·3	0·4	0·5	0·55	0·6	0·7	0·8	0·9	1·0
0·1	2·00	0·53	0·15	0·19	0·20	0·20	0·20	−0·09	−0·23	−0·24	−0·23	−0·21	−0·21
0·2	2·00	1·24	0·72	0·38	0·38	0·39	0·40	0·07	−0·18	−0·48	−0·58	−0·58	−0·55

distance $\frac{4}{3}$ of the total length *ahead* of the front. (This is because, as the table shows, the pressures producing lift are concentrated at the front while those at the rear produce negative lift.)

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THE FORMATION AND ENLARGEMENT OF A CIRCULAR HOLE IN A THIN PLASTIC SHEET

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SUMMARY

When a circular hole is made in a flat sheet by a conical-headed bullet or by outward radial pressure on its edge, the metal near the hole piles up into a thickened crater. The mechanics of this deformation is discussed. The interest of the problem lies in the fact that the complete strain-history of each element of the sheet has to be calculated. This is because the ratios of the principal stresses at each element of the sheet vary as the deformation proceeds, so that there is no relationship between the stress and total deformation but only between stress and strain increments occurring during a small expansion of the hole.

If b is the radius of the hole at any time, the strain is found to be elastic at points where r, the radius, is $> 3{\cdot}64b$. In the annulus $2{\cdot}21b < r < 3{\cdot}64b$ the strain is plastic, but comparable in magnitude with the small elastic strain when $r > 3{\cdot}64b$. In the annulus $b < r < 2{\cdot}21b$ there is finite strain. At the edge of the hole the sheet has thickened to $2{\cdot}61$ times the thickness of the sheet.

Experiments made with lead show that the symmetrical deformation contemplated in this analysis does not occur; but an alternative unsymmetrical deformation is produced which calculation shows to require less work, in the ratio $2{\cdot}6$ to $1{\cdot}0$, than the symmetrical mode.

Introduction

It is a comparatively easy matter to formulate simple laws of plasticity, but it is difficult to solve even the simplest special problems. Perhaps the most difficult feature in solving such problems is to connect the stress with the total deformation when the ratios of the principal stresses at each element of the material are not constant during the straining process. Most of the problems to which correct solutions have been given are concerned with plane strain of incompressible material, and in that case this difficulty does not arise. The chief interest in the solution of the special problem here discussed is that the ratios of the principal stresses change during the straining so that the whole history of the straining of each element has to be followed from the beginning to the end of the straining period.

In only one previously published case has this difficulty been faced.† The present solution was obtained before that work was started, and as its authors pointed out, their recognition of the fundamental difficulty in

† R. Hill, E. H. Lee, S. J. Tupper, 'The Theory of combined Plastic and Elastic deformation with particular reference to a thick tube under internal pressure', *Proc. Roy. Soc.* A, **191** (1947), 278.

this kind of work was prompted by some unpublished work of mine which they had seen. The solution which follows is part of that work. The remainder will not be published as it has been superseded by the much more complete work of Hill, Lee, and Tupper.

Enlargement of a hole in a sheet

When a hole is enlarged in a sheet of metal by radial pressure, as it is when pierced by a pointed conical broach or a pointed bullet, the metal near the edge of the hole is thickened over an area extending out to radius, say, r_2. Beyond this radius the metal suffers only small thickening, and outside some radius r_1 it suffers only elastic strain.

A simplified analysis illustrating the main features of this state of affairs was given in a private communication to the author by H. A. Bethe. With his permission a shortened version of this analysis is given below.

Bethe's analysis

The following assumptions were made:

1. The sheet is thin and the radius of the hole is large compared with the thickness. The sheet is of infinite extent.
2. Outside a radius r_1, the sheet is in a state of elastic stress due to radial pressure at r_1.
3. The stress σ_z perpendicular to the sheet is zero.
4. In a region $r_2 < r < r_1$ the plastic strain is small so that the stress distribution adjusts itself till a flow condition
$$\sigma_\theta - \sigma_r = Y \tag{1}$$
is fulfilled. Here Y is the yield stress.
5. Mohr's condition of plastic flow is satisfied so that the maximum difference in absolute value between the principal stresses is equal to Y. It is further assumed that Y is constant for large as well as small strains.

Using the equations of equilibrium and elasticity with assumptions (2) and (3), the following expressions for the stresses in the elastic region $r > r_1$ are derived:

$$\left.\begin{array}{ll} \text{radial stress} & \sigma_r = -A/r^2 \\ \text{tangential stress} & \sigma_\theta = +A/r^2 \\ \text{stress normal to sheet} & \sigma_z = 0 \end{array}\right\} \quad (r > r_1). \tag{2}$$

Here A is a constant for any given state of stress but varies as the straining proceeds.

Using the equation of equilibrium and the plasticity condition (4)

$$\left.\begin{aligned} \sigma_r &= Y\left(\ln\frac{r}{r_1}-\frac{1}{2}\right) \\ \sigma_\theta &= Y\left(\ln\frac{r}{r_1}+\frac{1}{2}\right) \\ \sigma_z &= 0 \end{aligned}\right\} \quad (r_1 > r > r_2). \tag{3}$$

The condition that σ_r is continuous at r_1 requires

$$\frac{A}{r_1^2} = \tfrac{1}{2}Y. \tag{4}$$

Bethe pointed out that, if Mohr's condition (5) is assumed, equations (1) and (3) are only applicable so long as σ_θ is positive (σ_r is necessarily negative), for it is only then that the maximum difference between the principal stresses is that between σ_r and σ_θ. When $\ln(r/r_1) < -\frac{1}{2}$ the maximum difference is between σ_r and σ_z. The radius r_2 at which $\sigma_\theta = 0$ is therefore

$$r_2 = r_1 e^{-\frac{1}{2}} = 0{\cdot}606 r_1,$$

or

$$r_1 = 1{\cdot}65 r_2. \tag{5}$$

Within the radius r_2 the condition of maximum stress difference could still be satisfied if $\sigma_\theta = 0$ and $\sigma_r = -Y$. The equilibrium equation applicable to sheets of uniform thickness, namely,

$$\frac{d\sigma_r}{dr}+\frac{\sigma_r-\sigma_\theta}{r} = 0, \tag{6}$$

would not be satisfied. On the other hand, if the material were able to flow within the radius r_2 in such a way that the thickness h were equal to $h_0 r_2/r$, h_0 being the original thickness, the equation of equilibrium of a thin sheet of variable thickness, namely,

$$\frac{d}{dr}(h\sigma_r)+\frac{h}{r}(\sigma_r-\sigma_\theta) = 0, \tag{7}$$

would be satisfied.

If the hole is expanded from a pin-hole to radius b and if the compressibility of the material be neglected, the volume of the metal contained between $r = b$ and $r = r_2$ must be equal to $\pi r_2^2 h_0$. Thus

$$\pi r_2^2 h_0 = \int_b^{r_2} \frac{h_0 r_2}{r}(2\pi r\, dr),$$

which yields

$$b = \tfrac{1}{2} r_2. \tag{8}$$

Hence

$$[h]_{r=b} = 2h_0. \tag{9}$$

Bethe's model indicates a reason for the main feature of the observed plastic strains round the hole made in a sheet by a conical-headed punch or bullet, namely, the thickening of the plate close to the hole. When, however, actual plastic strain near the hole is considered in detail it will be seen that the arbitrary assumption that $\sigma_\theta = 0$ and $\sigma_r = -Y$ is inconsistent with any theory of plasticity in which the ratios of increments of strain are related to corresponding ratios of stress differences. If $\sigma_\theta = \sigma_z = 0$ and $\sigma_r = -Y$ through the range $b < r < r_2$, any theory of stress and strain in an isotropic plastic material would require that the small increments in strain occurring while the radius of the hole was increased from b to $b+\delta b$ would be such that the radial strain e_r is equal to the normal strain e_z. It is clear that this is not consistent with the actual strain involved in the equation $h = h_0 r_2/r$. In fact, for a small increment δb in radius of the hole, Bethe's model involves a radial displacement δu, where

$$\delta u = \left(2-\frac{r}{b}\right)\delta b,$$

and the components of strain increment are

$$\left.\begin{aligned} \delta e_\theta &= \frac{2b-r}{r}\frac{\delta b}{b} \\ \delta e_z &= \frac{\delta h}{h} = \frac{4b}{r}\left(1-\frac{b}{r}\right)\frac{\delta b}{b} \\ \delta e_r &= \left(1-\frac{6b}{r}+\frac{4b^2}{r^2}\right)\frac{\delta b}{b} \end{aligned}\right\}. \tag{10}$$

Thus in Bethe's analysis the ratios $\delta e_r : \delta e_\theta : \delta e_r$ vary with radius while the ratios of the principal stresses are constant.

Problems in plasticity when the ratios of principal stress differences are not constant at each point during the straining process

The incompatibility, in any rational theory of plasticity, of Bethe's hypothesis ($\sigma_\theta = \sigma_z = 0$) with the strains represented by (10) must mean that this hypothesis is incorrect. To improve the analysis it is necessary to use some hypothetical assumption or experimental result relating the stress-difference ratios to the strain ratios. For small strains the simplest is that of Mises, namely,

$$\frac{\sigma_r-\sigma_\theta}{e_r-e_\theta} = \frac{\sigma_\theta-\sigma_z}{e_\theta-e_z} = \frac{\sigma_z-\sigma_r}{e_z-e_r}. \tag{11}$$

This, though not quite an accurate representation of experimental results,

is the hypothesis most frequently used in discussing problems in plasticity. It will be used in the work which follows.

If the stress-difference ratios, namely, $\sigma_r-\sigma_\theta:\sigma_\theta-\sigma_z:\sigma_z-\sigma_r$, are constant at each element of the material during the whole straining process, the corresponding strain-difference ratios $e_r-e_\theta:e_\theta-e_z:e_z-e_r$ are constant and (11) will apply if e_r, e_θ, and e_z are the total strains. If, however, the stress-difference ratios change in the course of the deformation, (11) can only apply if e_r, e_θ, and e_z are small increments of strain occurring during a portion of the deformation in which there is small change in the stress distribution. Problems of plastic flow in which the stress ratios vary during the deformation can therefore only be solved by following the strain-history of each element.

Eminent authorities have not always appreciated this point and have consequently published erroneous solutions of problems in which the stresses and *total* plastic strains have been related as though the stress distribution had been constant during the deformation when in fact it had not. The solutions of the problem of the straining of a thick-walled tube beyond the elastic limit which are given by Nadai† and by Sokolowsky‡ seem to me to be defective for this reason.

In the following pages the solution is given to the problem presented by a hole in a sheet of plastic material which is expanded from a pin-hole. The solution involves tracing the complete strain-history of each element of the sheet, but the analysis is much simplified by considerations of symmetry and similarity.

Analysis of strain round an expanding radial hole in a sheet

When a hole is enlarged the finite strain at any stage is made up of infinitesimal elements of strain which vary as the enlargement proceeds. Thus, when a small pin-hole in a plate is enlarged we must study the small strain produced in an element of the sheet which was originally at radius s from the pin-hole, when the hole enlarges from radius b to radius $b+\delta b$. In the more general case when the initial radius of the hole in the unstretched sheet is not zero this is very difficult to analyse, but when the expansion starts from a small pin-hole it may be expected that the configuration when the hole has radius b_2 will be similar to that round the hole when its radius is b_1 except that the radii where any given thickness occurs will be changed in the ratio b_2/b_1. Thus, if h is the thickness and u the radial displacement, it may be assumed that h/h_0 and u/b and also the stresses are functions of s/b only, where h_0 is the initial thickness of the sheet, and s the initial radial distance of a particle.

† *Plasticity* (McGraw Hill, 1931), pp. 196–9.

‡ *The Theory of Plasticity* (Russian) (Moscow, 1946), Chapter III.

To simplify matters I have assumed that the compressibility is so small that it may be neglected and the material taken as incompressible. The relationship between the small strain which occurs at any radius during the expansion of the hole through a small increase in radius from b to $b+\delta b$ can be understood by referring to Fig. 1. Here the ordinates represent u and the abscissae r.

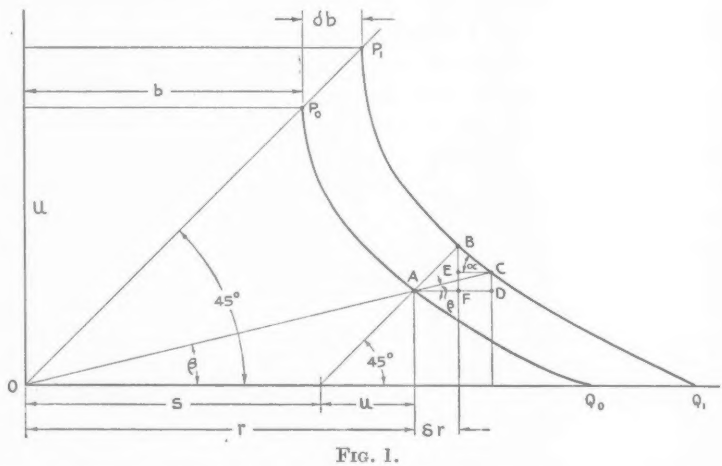

FIG. 1.

The initial radial distances of the element, which at a subsequent stage in the opening-out of the hole is at radius r, is related to u by the equation

$$r = s+u. \tag{12}$$

In Fig. 1, therefore, the displacement of a particle from its initial radius s is represented by a line drawn at 45° to the axes. In particular the displacement of the particles which were initially at the pin-point where the hole began is represented by the 45°-line OP_0P_1. The curved line P_0AQ_0 represents the relationship between r and u which it is the object of the analysis to calculate. At a subsequent stage of the expansion, when the hole has expanded from radius b to radius $b+\delta b$, the curve P_1BCQ_1 representing displacement is similar to P_0AQ_0, but with its linear dimensions increased in the ratio $b+\delta b:b$; thus in Fig. 1,

$$\frac{P_1P_0}{OP_0} = \frac{AC}{AO} = \frac{AD}{r} = \frac{\delta b}{b}$$

so that

$$AD = r\,\delta b/b. \tag{13}$$

If δr is the change in r *for a given particle of material* when the hole expands from b to $b+\delta b$, δr is found by drawing the line AB at 45° to the

axes to meet the curve $P_1 BCQ_1$ in B. If $\delta b/b$ is small enough, the arc CB may be taken as straight so that, if $\pi-\alpha$ is the slope of CB to the axis,

$$\frac{\partial u}{\partial r} = -\tan\alpha. \tag{14}$$

If β is the angle AOQ_0, $\tan\beta = u/r$. From the geometry of the figure $ABCD$ (Fig. 1)

$$\delta r = AF = BF = CE\tan\alpha+DA\tan\beta = (DA-\delta r)\tan\alpha+DA\tan\beta. \tag{15}$$

Hence
$$\delta r = \left(\frac{\tan\alpha+\tan\beta}{1+\tan\alpha}\right)DA, \tag{16}$$

and, from (13),
$$\delta r = \left(\frac{u/r-\partial u/\partial r}{1-\partial u/\partial r}\right)r\frac{\delta b}{b}. \tag{17}$$

The radial strain component during the expansion of the hole from b to $b+\delta b$ is $\frac{\partial}{\partial r}(\delta r)$, and differentiating (17) with respect to r, keeping δb constant,

$$\frac{\partial}{\partial r}(\delta r) = -\left[\frac{(1-u/r)r}{(1-\partial u/\partial r)^2}\frac{\partial^2 u}{\partial r^2}\right]\frac{\delta b}{b}. \tag{18}$$

Since the strain during expansion of the hole from b to $b+\delta b$ is proportional to $\delta b/b$, it is convenient to define strain components ϵ_r, ϵ_θ, and ϵ_z so that strains during the small enlargement δb are $\epsilon_r\,\delta b/b$, $\epsilon_\theta\,\delta b/b$, $\epsilon_z\,\delta b/b$. With this definition

$$\epsilon_r = -\left[\frac{r-u}{(1-\partial u/\partial r)^2}\right]\frac{\partial^2 u}{\partial r^2}. \tag{19}$$

The tangential strain is simply

$$\epsilon_\theta = \frac{b}{\delta b}\frac{\delta r}{r} = \frac{u/r-\partial u/\partial r}{1-\partial u/\partial r}, \tag{20}$$

and the strain perpendicular to the sheet is

$$\epsilon_z = -\epsilon_r-\epsilon_\theta. \tag{21}$$

The thickness h at any stage can be found simply from the equation of continuity: it is given by

$$\frac{h}{h_0} = \left(1-\frac{u}{r}\right)\left(1-\frac{\partial u}{\partial r}\right). \tag{22}$$

It can be verified that (21) is consistent with (22).

These expressions for strain take simple forms when expressed in terms of a new independent variable $\xi = r^2$ and a new dependent variable $\eta = s^2 = (r-u)^2$. Making these transformations and writing

$$p = \frac{d\eta}{d\xi}, \qquad q = \frac{d^2\eta}{d\xi^2}, \tag{23}$$

(19), (20), and (21) become

$$\epsilon_r = -1+\frac{2\eta q}{p^2}+\frac{\eta}{\xi p}, \tag{24}$$

$$\epsilon_\theta = 1-\frac{\eta}{\xi p}, \tag{25}$$

$$\epsilon_z = -\frac{2\eta q}{p^2}, \tag{26}$$

while (22) reduces to the simple form

$$h/h_0 = p. \tag{27}$$

(26) can be deduced directly from (27).

The stress-equilibrium equation for a thin sheet is

$$\frac{\partial}{\partial r}(h\sigma_r)+\frac{h(\sigma_r-\sigma_\theta)}{r} = 0. \tag{28}$$

Two possible alternative forms for the strength condition might be considered:

(*a*) Mohr's stress criterion which may be written

$$\begin{aligned} &\sigma_\theta-\sigma_r = Y \quad \text{if } \sigma_\theta \text{ is positive, i.e. tensile,} \\ \text{or} \quad &-\sigma_r = Y \quad \text{if } \sigma_\theta \text{ is negative, i.e. compressive.} \end{aligned} \tag{29}$$

(*b*) Mises's condition which may be written, when $\sigma_z = 0$,

$$\sigma_r^2+\sigma_\theta^2-\sigma_\theta\,\sigma_r = \text{constant}. \tag{30}$$

This reduces to $-\sigma_r = \text{constant}$ if $\sigma_\theta = 0$, and so is identical with Mohr's in that case.

If Bethe's assumption that $\sigma_\theta = 0$ combined with $\sigma_r = \text{constant}$ is used, (28) leads to

$$hr = \text{constant} = h_0 r_2', \tag{31}$$

where r_2' is the outer boundary of the region of finite plastic strain. Substituting in (22)

$$\frac{r_2'}{r} = \left(1-\frac{u}{r}\right)\left(1-\frac{\partial u}{\partial r}\right), \tag{32}$$

which gives on integration

$$\tfrac{1}{2}(r-u)^2 = rr_2'+\text{constant}. \tag{33}$$

Since $u = 0$ when $r = r_2'$, the constant is $-\frac{1}{2}(r_2')^2$ and

$$u = r-\sqrt{\{(2r-r_2')r_2'\}}. \tag{34}$$

The inner boundary is where $b = r = u$, so that from (34)

$$b = \tfrac{1}{2}r_2', \tag{35}$$

which is Bethe's result if r_2' is identified with his r_2.

Introduction of the strain-ratio relationship

Since the strain-ratio relationship (11) involves only differences between the principal stresses, we may assume without loss of generality that $\sigma_z = 0$. We shall assume further that the compressibility of the material is small enough to be neglected so that $\epsilon_r+\epsilon_\theta+\epsilon_z = 0$. In these circumstances (11) may be written

$$\frac{\sigma_\theta}{\sigma_r} = \frac{\epsilon_\theta-\epsilon_z}{\epsilon_r-\epsilon_z} = \frac{\epsilon_r+2\epsilon_\theta}{2\epsilon_r+\epsilon_\theta}. \tag{36}$$

Substituting (27) and (36) in (28) the equilibrium condition reduces to

$$2\frac{d}{d\xi}(p\sigma_r)+\frac{p\sigma_r}{\xi}\left(\frac{\epsilon_r-\epsilon_\theta}{2\epsilon_r+\epsilon_\theta}\right) = 0. \tag{37}$$

This equation must be used in conjunction with a strength criterion. Mohr's criterion (*a*) will be used. In this case (37) assumes two different forms according as σ_θ is negative (i.e. compressive) or positive (i.e. tangential tension). These are:

σ_θ negative, $\sigma_r = -Y$, so that (37) becomes

$$2q+\frac{p}{\xi}\left(\frac{\epsilon_r-\epsilon_\theta}{2\epsilon_r+\epsilon_\theta}\right) = 0; \tag{38}$$

σ_θ positive, $\sigma_r-\sigma_\theta = -Y$, so that, from (36),

$$\sigma_r = \frac{2\epsilon_r+\epsilon_\theta}{\epsilon_r-\epsilon_\theta}(-Y)$$

and hence

$$2\frac{d}{d\xi}\left[p\left(\frac{2\epsilon_r+\epsilon_\theta}{\epsilon_r-\epsilon_\theta}\right)\right]+\frac{p}{\xi} = 0. \tag{39}$$

Substituting for ϵ_r and ϵ_θ from (24) and (25) the resulting equations may be written:

σ_θ *negative* (*tangential compression*)

$$q^2\left(\frac{4\eta}{p^2}\right)+q\left(-1+\frac{2\eta}{\xi p}\right)+\frac{p}{\xi}\left(-1+\frac{\eta}{\xi p}\right) = 0. \tag{40}$$

In this case, from (38),

$$\frac{\epsilon_\theta}{\epsilon_r} = \frac{4q+p/\xi}{-2q+p/\xi}, \qquad \frac{\sigma_\theta}{\sigma_r} = 1-\frac{2q\xi}{p}, \tag{41}$$

and in terms of Mohr's strength criterion the stresses are

$$\sigma_r = -Y, \qquad \sigma_\theta = -Y\left(\frac{\sigma_\theta}{\sigma_r}\right). \tag{42}$$

σ_θ *positive (tangential tension)*

$$3w\left(\frac{\eta}{p^2}\right)\left(p-\frac{\eta}{\xi}\right) = \frac{4\eta^2q^3}{p^4}+q^2\left(\frac{3\eta^2}{\xi p^3}+\frac{\eta}{p^2}\right)+2q\left(-1+\frac{3\eta^2}{\xi^2p^2}-\frac{2\eta}{\xi p}\right)+\frac{p}{\xi}-\frac{2\eta}{\xi^2}+\frac{\eta^2}{\xi^3 p}, \tag{43}$$

where w is written for $\dfrac{dq}{d\xi}$, i.e. $\dfrac{d^3\eta}{d\xi^3}$.

The expressions for $\epsilon_\theta/\epsilon_r$ and σ_θ/σ_r cannot be simplified by using the equation of equilibrium, and the full expressions derived from (24), (25), and (36) must be used, namely,

$$\frac{\epsilon_\theta}{\epsilon_r} = \frac{1-\eta/\xi p}{-1+2\eta q/p^2+\eta/\xi p}, \qquad \frac{\sigma_\theta}{\sigma_r} = \frac{6\eta q}{4\eta q+\eta p/\xi-p^2}-1, \tag{44}$$

and in terms of Mohr's condition $\sigma_r-\sigma_\theta = -Y$ the stresses are now

$$\sigma_r = -Y/(1-\sigma_\theta/\sigma_r), \qquad \sigma_\theta = -Y\left(\frac{\sigma_\theta}{\sigma_r}\right)\Big/\left(1-\frac{\sigma_\theta}{\sigma_r}\right). \tag{45}$$

It will be seen that (43) is an ordinary differential equation of the third order and first degree while (40) is of the second order and second degree. The reason for this difference lies in the form of Mohr's strength condition. When σ_θ is positive three boundary conditions can be assigned at any given value of ξ (i.e. of r). These might, for instance, be u/r, h/h_0, and σ_r which can be transformed directly in assigned values of q, p, and η. When σ_θ is negative σ_r cannot be assigned arbitrarily; it is in fact constant. Thus only p and η can be assigned arbitrarily.

Boundary condition at the elastic–plastic boundary

The elastic stresses due to radial displacement in an infinite sheet are

$$-\sigma_r = \sigma_\theta = \tfrac{1}{2}Yr_1^2/r^2, \tag{46}$$

where r_1 is the radius at which $\sigma_r-\sigma_\theta = -Y$. The corresponding small radial displacement is

$$u = \frac{1+m}{2}\left(\frac{Y}{E}\right)\frac{r_1^2}{r}, \tag{47}$$

where E is Young's modulus and m is Poisson's ratio. In the present investigation compressibility will be neglected and we will take $m = \frac{1}{2}$. In the elastic region, therefore, where u/r is small compared with unity,

$$\eta = r^2\left(1-\frac{u}{r}\right)^2 = \xi\left(1-\frac{3}{2}\frac{Y}{E}\frac{r_1^2}{\xi}\right). \tag{48}$$

At the inner boundary of the elastic region therefore

$$p = \frac{d\eta}{d\xi} = 1, \qquad q = 0, \qquad \eta = \xi\left(1-\frac{3Y}{2E}\right). \tag{49}$$

At the outer boundary of the plastic region, since σ_r is positive, Mohr's criterion ensures that $\sigma_r - \sigma_\theta = -Y$. Since σ_r is necessarily continuous through $r = r_1$, and it is assumed that $\sigma_r - \sigma_\theta = -Y$ at the elastic limit in the elastic region, σ_θ must be continuous through $r = r_1$ and equal to $\frac{1}{2}Y$. It is important to notice the reason why σ_θ is continuous at the plastic boundary in this case, because it is not necessary in general that σ_θ shall be continuous when Mohr's criterion is used. It will be shown in fact that σ_θ becomes discontinuous on the circle $r = r_2$ within the plastic region at the point where $\sigma_\theta = 0$.

Strains and displacements when $r_1 > r > r_2$

In the region within the circle $r = r_1$, where σ_θ is positive, it will be found that the strains are small, being of order Y/E. Assuming that

$$\eta = \xi(1-\alpha\eta_1) \quad \text{and} \quad p = 1+\alpha p_1,$$

where

$$\alpha = 3Y/2E,$$

$$p = \frac{d\eta}{d\xi} = 1-\alpha\eta_1-\alpha\xi\frac{d\eta_1}{d\xi} \quad \text{so that} \quad p_1 = -\eta_1-\xi\frac{d\eta_1}{d\xi}, \tag{50}$$

and

$$q = \frac{dp}{d\xi} = \alpha\frac{dp_1}{d\xi} = -\alpha\left(2\frac{d\eta_1}{d\xi}+\xi\frac{d^2\eta_1}{d\xi^2}\right). \tag{51}$$

When σ_θ is positive

$$\sigma_r = -Y\left(\frac{2\epsilon_r+\epsilon_\theta}{\epsilon_r-\epsilon_\theta}\right) = -\tfrac{1}{2}Y\left(\frac{-p+\eta/\xi+4\eta q/p}{-p+\eta/\xi+\eta q/p}\right). \tag{52}$$

Substituting from (50) and (51) in (52) and neglecting terms in α^2 compared with those containing α,

$$\sigma_r = -\tfrac{1}{2}Y\left(4+\frac{3\psi}{\psi+\xi\psi'}\right),$$

$$\text{where} \quad \psi = \frac{d\eta_1}{d\xi} \quad \text{and} \quad \psi' = \frac{d\psi}{d\xi}. \tag{53}$$

Substituting this in (39),

$$\frac{d}{d\xi}\left(\frac{p\psi}{\psi+\xi\psi'}\right)+\frac{p}{3\xi} = 0. \tag{54}$$

Neglecting terms which contain α as a factor compared with those that do not, p may be taken as 1·0 and (54) may then be integrated giving

$$\frac{\psi}{\psi+\xi\psi'}+\tfrac{1}{3}\ln\xi = \text{constant}. \tag{55}$$

The boundary condition at $r = r_1$ is $\sigma_r = -\sigma_\theta = -\frac{1}{2}Y$ so that from (53) $\psi/(\psi+\xi\psi') = -1$. The constant in (55) is therefore $-1+\frac{1}{3}\ln r_1^2$. Writing ζ for $\ln(\xi/r_1^2)$, (55) becomes

$$\frac{d\psi}{d\zeta}+\psi\left(\frac{6+\zeta}{3+\zeta}\right) = 0. \tag{56}$$

The integral of (56) is

$$\ln\psi+\zeta+3\ln(3+\zeta) = \text{constant}. \tag{57}$$

Throughout the whole of the elastic region $-\sigma_r = \sigma_\theta$ and $\sigma_z = 0$, so that, at the elastic boundary, $p = 1$; hence, from (50),

$$\left[\eta_1+\xi\frac{d\eta_1}{d\xi}\right]_{r=r_1} = 0, \tag{58}$$

and from the definition of η_1, (49) shows that $[\eta_1]_{r=r_1} = 1$. Hence from (58), $[\xi\psi]_{r=r_1} = -1$; (57) may therefore be written

$$\psi\xi(1+\tfrac{1}{3}\ln\xi r_1^{-2})^{-3} = -1, \tag{59}$$

and since $\psi = \dfrac{d\eta_1}{d\xi}$ and $\zeta = \xi r_1^{-2}$ this becomes

$$\frac{d\eta_1}{d\zeta} = -\frac{1}{(1+\frac{1}{3}\zeta)^3}. \tag{60}$$

(60) may be integrated, giving

$$\eta_1 = \frac{3}{2(1+\frac{1}{3}\zeta)^2}-\frac{1}{2}. \tag{61}$$

The equation for the thickness is

$$\frac{h}{h_0} = \frac{d\eta}{d\xi} = 1-\alpha\left(\eta_1+\frac{d\eta_1}{d\xi}\right) = 1+\alpha\left[\frac{1}{2}-\frac{1+\zeta}{2(1+\frac{1}{3}\zeta)^3}\right]. \tag{62}$$

The displacement is

$$u = \xi^{\frac{1}{2}}-\eta^{\frac{1}{2}} = \tfrac{1}{2}\alpha\eta r = \tfrac{1}{2}\alpha r[\tfrac{3}{2}(1-\tfrac{1}{3}\zeta)^{-2}-\tfrac{1}{2}]. \tag{63}$$

The stress can be found by substituting from (56) $-\dfrac{6+\zeta}{3+\zeta}$ for $\dfrac{1}{\psi}\dfrac{d\psi}{d\zeta}$, i.e. for $\dfrac{\xi\psi'}{\psi}$, in (53). It is found that

$$\sigma_r = -\tfrac{1}{2}Y(1-\zeta) = -\tfrac{1}{2}Y\{1-2\ln(r/r_1)\}, \tag{64}$$

and hence

$$\sigma_\theta = Y-\sigma_r = \tfrac{1}{2}Y\left(1+2\ln\frac{r}{r_1}\right). \tag{65}$$

This is the well-known result which can be obtained without considering the strains and displacements, if it is assumed that the thickness of the plate does not vary.

Values when $\sigma_\theta = 0$

The radius r at which $\sigma_\theta = 0$ is from (50) $r_2 = r_1 e^{-\frac{1}{2}} = 0{\cdot}606r_1$ and corresponds to $\zeta = -1$.

Though the stress distribution in the range $r_1 > r > r_2$ is identical with that found in Bethe's investigation, the displacements and strains are not

the same. For the case when Poisson's ratio is $\frac{1}{2}$ Bethe's method would give for the displacement when $\sigma_\theta = 0$

$$u_2 = \tfrac{1}{2}\alpha\frac{r_1^2}{r_2} = \tfrac{1}{2}\alpha r_2(2\cdot718). \tag{66}$$

Putting $\zeta = -1$ in (63) the displacement according to the present strain hypothesis is

$$u_2 = \frac{\alpha r_2}{2}\left\{\frac{3}{2(\frac{2}{3})^2}-\frac{1}{2}\right\} = \tfrac{1}{2}\alpha r_2\left(\frac{23}{8}\right) = \tfrac{1}{2}\alpha r_2(2\cdot875). \tag{67}$$

The displacement is in fact about 6 per cent. greater than that calculated on Bethe's strain hypothesis.

Putting $\zeta = -1$ in (62) the value of h/h_0 at $r = r_2$ is $1+\frac{1}{2}\alpha$, and from (51) and (56)

$$q = -\alpha\left(2\psi+\frac{d\psi}{d\zeta}\right) = -\alpha\psi\left(2-\frac{6+\zeta}{3+\zeta}\right) = -\alpha\psi\left(\frac{\zeta}{3+\zeta}\right);$$

hence, from (57),
$$q\xi = \frac{\alpha\zeta}{(3+\zeta)(1+\frac{1}{3}\zeta)^3},$$

so that when $\zeta = -1$,
$$q\xi = -\tfrac{27}{16}\alpha. \tag{68}$$

Boundary values at $\sigma_\theta = 0$

At the circle $r = r_2$, where $\sigma_\theta = 0$, p and η are continuous. Just inside the circle therefore, where σ_θ is negative,

$$\frac{\eta}{\xi} = 1-\alpha\eta_1 = 1-\tfrac{23}{8}\alpha \quad\text{and}\quad p = \frac{h}{h_0} = 1+\tfrac{1}{2}\alpha. \tag{69}$$

When σ_θ is negative q is determined by (40) when η/ξ and p are given. Substituting in (40) from (69), the values of $q\xi$ found by solving the resulting quadratic equation are (neglecting terms in α^2)

$$q\xi = -\tfrac{1}{4}-\tfrac{85}{32}\alpha \quad\text{and}\quad q\xi = +\tfrac{27}{8}\alpha. \tag{70}$$

Neither of these values is the same as $q = -27\alpha/16$, the value just outside the boundary, so that q is *not continuous* at $r = r_2$.

The two possible values of q just inside the radius $r = r_2$ are therefore of opposite signs. Since

$$q = \frac{dp}{d\xi} = \frac{1}{h_0}\frac{dh}{d\xi} = \frac{1}{2h_0 r}\frac{dh}{dr},$$

a positive value of q would correspond to a condition in which the plate gets thinner towards the centre, a configuration which does not appear to have physical significance. The negative value of q for which $q\xi = -\frac{1}{4}-\frac{85}{32}\alpha$ will therefore be taken as the boundary value to be used in continuing the

solution inward from $r = r_2$. It will be noticed that ϵ_r, and consequently σ_θ, are discontinuous as well as q. This discontinuity arises from the form of Mohr's criterion. It would not occur if von Mises's criterion had been used.

Discontinuity in ϵ_r and ϵ_θ

Substituting

$$q\xi = -\tfrac{1}{4}-\tfrac{85}{32}\alpha, \qquad p = 1+\tfrac{1}{2}\alpha, \qquad \frac{\eta}{\xi} = 1-\tfrac{23}{8}\alpha$$

in (24) and (25), it is found that

$$\epsilon_\theta = \tfrac{27}{8}\alpha, \qquad \epsilon_r = -\tfrac{1}{2}-\tfrac{27}{4}\alpha,$$

and, substituting these in (36),

$$\frac{\sigma_\theta}{\sigma_r} = \tfrac{1}{2}-\tfrac{81}{16}\alpha.$$

When α is small, i.e. when E/Y is small, we may neglect α and take as the boundary condition at $r = r_2$ for calculating the stresses and displacements when $r < r_2$ the values

$$p = 1, \qquad \eta/\xi = 1, \qquad q\xi = -\tfrac{1}{4} \tag{71}$$

and the stresses are $\sigma_r = -Y$, $\sigma_\theta = -\frac{1}{2}Y$.

Thus the stress σ_θ suddenly changes from 0 to a *compressive stress* of $-\frac{1}{2}Y$ at the radius $r = r_2$.

Calculation of stress and strain when $r < r_2$

To calculate the distribution of stress and plastic strain inside the radius $r = r_2$, (40) must be solved step by step using the boundary values (71).

If $\delta\xi$ is the magnitude of a small step, the corresponding changes p and η may be taken as

$$\left.\begin{aligned} \delta\eta &= p\,\delta\xi+\tfrac{1}{2}q(\delta\xi)^2 \\ \delta p &= q\,\delta\xi \end{aligned}\right\}. \tag{72}$$

After calculating the values of p and η at the end of each step these values are inserted in (40) and the resulting quadratic for q is solved, the root which derives by continuous variation of η, and p from $qr_2^2 = -\frac{1}{4}$, being chosen in each case.

The results of applying this process are given in Table 1 and are shown graphically in Fig. 2. Values of the principal variables ξ/r_2^2 and η/r_2^2 are given in cols. 1 and 2, Table 1. Values of p and $-qr_2^2$ are given in cols. 3 and 4. Using these values of p, q, and ξ, values of σ_θ/σ_r calculated from (41) are given in col. 8, and the corresponding values of σ_r/Y and σ_θ/Y in cols. 9 and 10. It will be seen that σ_θ, which begins as a compressive

stress equal to half the radial stress at the outer limit of the region of finite plastic flow, rapidly decreases till when $\xi/r_2^2 = 0{\cdot}35$ it becomes zero, and if the process is carried farther, using (41), σ_θ becomes a tension.

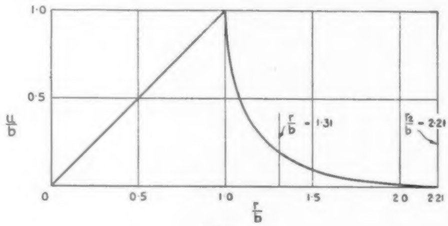

Fig. 2.

When $\xi/r^2 = 0{\cdot}30$, for instance, the calculated value of σ_θ/σ_r is $-0{\cdot}124$. For values of ξ/r_2^2 less than 0·35, therefore, the alternative form (43) of the equilibrium equation must be used.

Table 1

1	2	3	4	5	6	7	8	9	10	
ξ/r_2^2	η/r_2^2	p	$-qr_2^2$	wr_2^4	r/b	u/b	σ_θ/σ_r	σ_r/Y	σ_θ/Y	
1·0	1·0	1·0	0·25		2·21	0	+0·50	−1·0	−0·50	from eqn. (40), σ_θ negative.
0·90	0·899	1·025	0·305		2·096	·001		−1·0	−0·47	
0·80	0·795	1·055	0·381		1·978	·008	+0·440	−1·0	−0·44	
0·75	0·741	1·075	0·431		1·915	·012	+0·397	−1·0	−0·40	
0·70	0·687	1·096	0·493		1·850	·019	+0·370	−1·0	−0·37	
0·65	0·632	1·121	0·566		1·782	·024	+0·343	−1·0	−0·34	
0·60	0·575	1·149	0·660		1·712	·035	+0·310	−1·0	−0·31	
0·50	0·457	1·257	0·998		1·563	·070	+0·212	−1·0	−0·21	
0·45	0·392	1·317	1·240		1·483	·100	+0·152	−1·0	−0·15	
0·40	0·325	1·379	1·583		1·398	·138	+0·082	−1·0	−0·08	
0·35	0·254	1·450	2·070		1·307	·194	+0·000	−1·0	0	
0·30	0·179	1·554	2·910		1·210	·276	−0·124	−1·0	+0·12	
0·35	0·254	1·450	2·070	26·5	1·308	·194				from eqn. (43), σ_θ positive.
0·30	0·178	1·587	3·397	57·3	1·210	·278	−0·092	−0·916	+0·084	
0·27	0·129	1·715	5·117	108·0	1·149	·356	−0·168	−0·857	+0·143	
0·24	0·075	1·917	8·357	281·0	1·083	·478	−0·328	−0·753	+0·247	
0·22	0·034	2·140	13·98	934·0	1·037	·630	−0·569	−0·638	+0·362	
0·21	0·012	2·326	23·22	5190·0	1·013	·771	−0·739	−0·576	+0·424	
0·205	0	2·61			1·00	1·00	−1·000	−0·500	+0·500	

Since σ_r is continuous and equal to $-Y$ at $\xi/r_2^2 = 0{\cdot}35$ and $\sigma_\theta = 0$ when ξ/r^2 is just greater than 0·35, while Mohr's criterion ensures that

$$\sigma_r - \sigma_\theta = -Y$$

when σ_θ is positive, it seems that $\sigma_\theta = 0$ when ξ/r_2^2 is just less than 0·35. Since both σ_r and σ_θ are therefore in this case continuous through the radius where σ_θ changes sign, ϵ_r and ϵ_θ are also continuous. Hence from

(24) q is continuous. The values of η, p, and q at $\xi/r_2^2 = 0{\cdot}35$ can therefore be inserted in (43) and the value of $w = d^3\eta/d\xi^3$ at $\xi/r_2^2 = 0{\cdot}35$ determined.

The changes in η, p, and q during the first step $\delta\xi$ in the new region are calculated using the formulae

$$\begin{aligned} \delta\eta &= p\,\delta\xi + \tfrac{1}{2}q(\delta\xi)^2 + \tfrac{1}{6}w(\delta\xi)^3, \\ \delta p &= q\,\delta\xi + \tfrac{1}{2}w(\delta\xi)^2, \\ \delta q &= w\,\delta\xi. \end{aligned} \tag{73}$$

Values of η/r_2^2, p, $-qr_2^2$, and wr_2^4 found in this way are given in the lower part of Table 1, corresponding to $0{\cdot}35 > \xi/r_2^2 > 0{\cdot}205$. Values of σ_θ/σ_r, σ_r/Y, and σ_θ/Y from (44) and (45) are given in cols. 8, 9, 10 of Table 1.

Conditions at edge of hole

It will be seen in Table 1 that as ξ/r_2^2 decreases to $0{\cdot}21$, $-qr_2^2$ and wr_2^4 are rising very rapidly. A study of the values of the terms in (43) reveals that by the time $\xi/r_2^2 = 0{\cdot}21$ is reached, one term on the right-hand side of the equation and one on the left-hand side are larger than any other terms. The limiting form of the equation when η is small is in fact

$$\frac{3w\eta}{p} = -2q. \tag{74}$$

This equation can be integrated twice, thus

$$\left.\begin{aligned} -q &= A\eta^{-\frac{2}{3}} \\ \text{and} \qquad p^2 &= B - 6A\eta^{\frac{1}{3}} \end{aligned}\right\}, \tag{75}$$

A and B being the two constants of integration. To determine the constants the values of $\eta/r_2^2 = 0{\cdot}012$, $p = 2{\cdot}326$, $qr_2^2 = -23{\cdot}22$ can be used at $\xi = 0{\cdot}21$. The resulting values of A and B are

$$A = 1{\cdot}217r_2^{-\frac{2}{3}}, \qquad B = 7{\cdot}07. \tag{76}$$

The limiting value of p when $\eta = 0$ is therefore

$$p = \sqrt{7{\cdot}07} = 2{\cdot}66. \tag{77}$$

This is the limiting value of h/h_0 at the edge of the hole and may be compared with Bethe's value $2{\cdot}0$. It is not very different from the value at $\xi/r_2^2 = 0{\cdot}21$; to find the limiting value of ξ therefore it is sufficient to take p as constant and equal to $2{\cdot}61$ in the interval during which η decreases from $0{\cdot}012$ to 0. Thus the limiting value of ξ/r_2^2 corresponding to the edge of the hole is

$$\xi_{\eta=0}\,r_2^{-2} = 0{\cdot}21 - \frac{0{\cdot}012}{2{\cdot}66} = 0{\cdot}205. \tag{78}$$

The ratio $\left(\dfrac{\text{radius of finite plastic deformation}}{\text{radius of hole}}\right)$ is

$$\frac{r_2}{b} = \frac{1}{\sqrt{0{\cdot}205}} = 2{\cdot}21. \tag{79}$$

This may be compared with Bethe's value 2·0.

Substituting the approximate limiting forms of p and q from (75) in (44), the limiting form for σ_θ/σ_r is

$$\lim_{\eta\to 0}\left(\frac{\sigma_\theta}{\sigma_r}\right) = \lim_{\eta\to 0}\frac{7{\cdot}30\eta^{\frac{1}{3}}r_2^{-\frac{4}{3}}}{7{\cdot}07-2{\cdot}43\eta^{\frac{1}{3}}r_2^{-\frac{4}{3}}}-1. \tag{80}$$

This tends to the value -1 as indicated in the last figure of col. 8, and the corresponding values of σ_r and σ_θ are therefore $-0{\cdot}5Y$ and $+0{\cdot}5Y$.

It will be noticed that the stress at the internal boundary could have been predicted *a priori* if it had been possible to assume that h/h_0 is finite at $r = b$, because clearly the total amounts of strain in the tangential and radial directions are both infinite at a hole which has been enlarged from a pin-hole. Thus the state of strain at the hole is such that symmetry alone must ensure that σ_z is exactly half-way between σ_r and σ_θ. Since $\sigma_z = 0$, $\sigma_r = -\sigma_\theta$. Similar considerations can be used to understand why the stress at points just inside the boundary $r = r_2$ corresponds with $(\sigma_\theta/\sigma_r) = +0{\cdot}5$, for at the edge of the region of finite plastic displacement, where the radial displacement is zero, $\epsilon_\theta = 0$. Thus $\epsilon_r = -\epsilon_z$ and σ_θ must therefore be exactly half-way between σ_r and σ_z. Hence, since $\sigma_z = 0$, $\sigma_\theta = \frac{1}{2}\sigma_r$.

Expressions in terms of radius of hole

The radial variable is expressed in terms of the radius of the plastic region. To express the results in terms of b, it is necessary to tabulate $r/b = 2{\cdot}21\sqrt{\xi}/r_2$. These values are given in col. 6, Table 1. The displacements $u/b = 2{\cdot}21(\sqrt{\xi}-\sqrt{\eta})r_2^{-1}$ are tabulated in col. 7.

The radial displacement is shown graphically in Fig. 2 which may be compared with the diagrammatic sketch, Fig. 1.

Comparison with Bethe's results

The thickness ratio $h/h_0 = p$ is shown in Fig. 3 and Bethe's values, namely, $h/h_0 = 2b/r$, are also shown. It will be seen that the main differences are that the present calculation shows the 'crater' extending farther radially than Bethe's and at the same time the 'crater' is much steeper close to the hole. From (75) it will be seen that $q \to -\infty$ as $\eta \to 0$ and, since $q = \dfrac{1}{2h_0 r}\dfrac{dh}{dr}$, the equations indicate that the edge of the 'crater' is

a thin knife-edge. This deduction, however, is based on the assumption that the strains and stresses are uniform through the thickness of the plate. It is evident that this assumption ceases to be a good approximation when $\frac{dh}{dr} \to -\infty$. At first sight it might be thought that the extra thickness at $r = b$ above Bethe's $2h$ means that the work done in expanding the hole

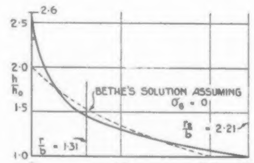

FIG. 3.

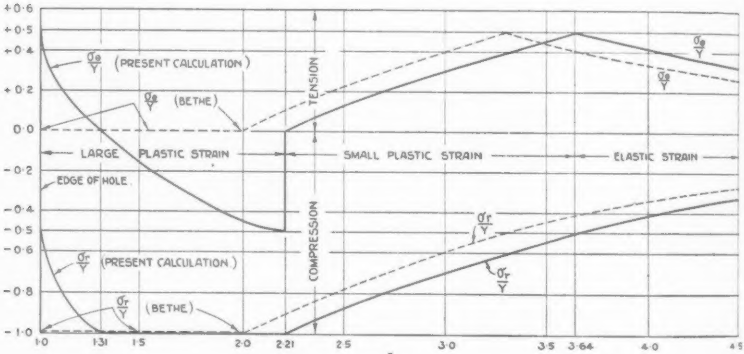

FIG. 4.

is greater according to the present calculations than in Bethe's calculations, but this is the reverse of the truth, for the radial stress at the hole is only $-\frac{1}{2}Y$ instead of Bethe's $-Y$. In fact the work done in expanding to a given radius is

$$\pi b^2 h_0(\tfrac{1}{2}Y)(2{\cdot}66) = 1{\cdot}33\pi b^2 h_0 Y \tag{81}$$

while if Bethe's strain assumption is used it is $2\pi b^2 h_0 Y$.

Fig. 4 shows the distribution of stress. This is of course very different from Bethe's, the most striking difference being that the present calculations predict a state of tangential tension in a ring which extends to 30 per cent. of the radius of the hole from its edge and a tangential compression from that point to the edge of the region of large plastic

distortion. In the plastic region $r_1 > r > r_2$, where small strains comparable with the elastic strains occur, the stress is as calculated by Bethe, i.e. there is a tangential tension. In this connexion it may be noticed that in comparing calculations of this kind with the behaviour of real materials, a metal which experiences considerable hardening with cold work might give results differing widely from the above theory.

Experimental work

An attempt was made to produce the distribution of strain contemplated in the foregoing analysis. A series of tapering sections of a cone of semi-vertical angle $1\frac{1}{2}°$ were made in mild steel. The smallest of these, which terminated in the vertex, was mounted in the chuck of a lathe and a flat sheet of lead was mounted on a slide rest so that its plane was perpendicular to the lathe bed. A small hole was bored in the lead sheet and the slide rest was moved parallel to the axis of rotation of the cone so that the vertex entered the hole and exerted radial pressure on its circumference. The cone was rotated while the hole was being enlarged so that the direction of the friction should be in the tangential direction rather than perpendicular to the plane of the lead sheet. 'Oildag' was used as a high-pressure lubricant between the sheet and the cone.

On reaching the base of the smallest section the lead sheet was withdrawn and the next largest section of the cone was fitted in the chuck. The hole was then enlarged to the largest diameter of this section. This process was repeated with successive sections, or broaching tools, till the hole was 2·45 cm. in diameter.

It was found that the hole could be enlarged symmetrically without bending the plate till its diameter was between 7 and 10 times the thickness of the sheet, but that at about this stage the sheet always bent out of its plane. This was not due to a sideways pressure, for the sheet sometimes bent towards the thick end of the broaching tool and sometimes away from it. It appears that the sheet is unstable and that an alternative form of deformation occurs in which the 'crater' develops into a short length of tube on one side or other of the sheet.

This tube is joined to the flat sheet by a region where the sheet is bent into the form of a curved fillet. The wall of the tube varies in thickness from that of the sheet at the fillet to zero at the outer edge. This edge contains the particles which were originally at the point where the vertex of the cone entered the sheet.

Sections of deformed sheets are shown in Fig. 5. These photographs were obtained by sawing the sheets in two in planes passing through their axes of symmetry and grinding down the surfaces so obtained till the true

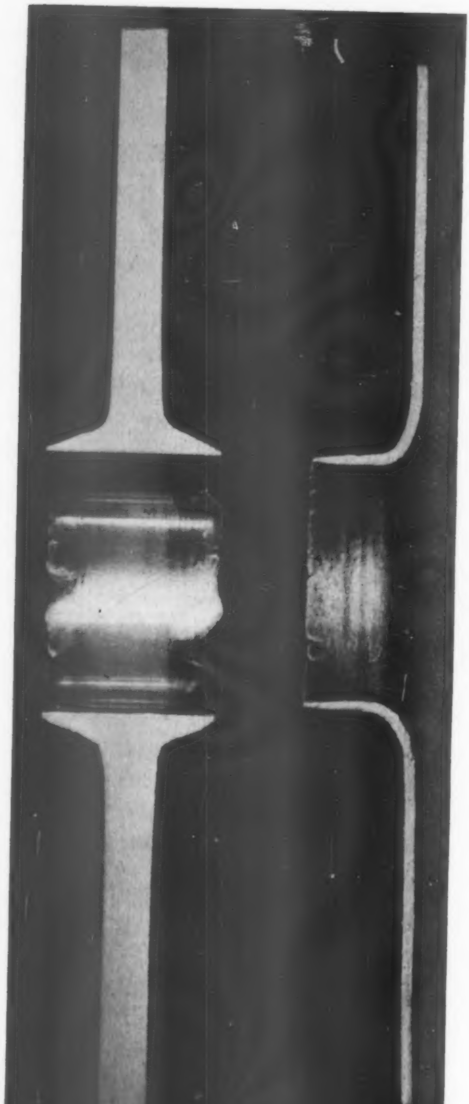

FIG. 5.

sections were revealed. The upper photograph shows the hole when its radius is 2·7 times the thickness of the sheet, while in the lower photograph the hole is about 10 times the thickness. It is clear that in the unsymmetrical case (lower photograph, Fig. 5) the material of the tube-shaped crater is expanding under the influence of a tangential tension Y and that the stress perpendicular to the plane of the sheet is zero. The radial stress varies from zero at the outer surface to approximately Yt/b at its inner surface. Here b is the radius of the hole and t the thickness of the wall of the tubular crater. Since t is less than h_0 the radial stress is negligible when b/h_0 is large. The stress is therefore approximately uni-axial. The contraction of the material in the radial direction should therefore be equal to the contraction in the direction perpendicular to the sheet. This type of deformation would give rise to a distribution of thickness t in the wall of the tubular crater for which $t = h_0\sqrt{(y/D)}$, where y is the distance from the plane of the lip of the crater and D is the height of this plane above the plane of the sheet. The initial radius, s, of the ring of particles which are at the point in the crater defined by y is given by

$$\frac{s}{b} = \sqrt{\frac{y}{D}}. \tag{82}$$

Since the material is assumed to be incompressible

$$\pi b^2 h_0 = 2\pi b \int_0^D \sqrt{\frac{y}{D}}\,dy, \tag{83}$$

so that

$$D = \tfrac{3}{4}b \tag{84}$$

and

$$t = h_0\sqrt{\frac{y}{0{\cdot}75b}} = 1{\cdot}15h_0\sqrt{\frac{y}{b}}. \tag{85}$$

The main features of the approximate theory represented by (84) and (85) are found, at any rate qualitatively, in the experiment represented in the lower photograph of Fig. 5. Since this unsymmetrical mode of deformation and also the symmetrical deformation are both theoretically possible it is of interest to compare the work done in opening out the hole to a given radius b by the two modes.

The work done in expanding the ring of material which was originally contained between radii s and $s+\delta s$ is

$$2\pi Y h_0 s\,\delta s \ln\frac{b}{s}$$

so that the total work done is

$$w = 2\pi h_0 Y \int_0^b s \ln\frac{b}{s}\, ds = \tfrac{1}{2}\pi b^2 h_0 Y. \tag{86}$$

Comparing (86) with (81), it will be seen that the symmetrical mode requires 2·6 times as much work as the unsymmetrical mode for the same radius of hole. This fact seems to explain why it is the unsymmetrical type of deformation which actually occurs.

(86)

mode
same
etrical